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ABSTRACT

The purpose of this text is to help elementary school teachers achieve balance in the teaching of mathematics. Children must acquire (1) computational skills, (2) conceptual ideas, and (3) knowledge of applications of mathematics. The text provides reading materials for the teacher as well as problems and exercises to help fix the ideas in mind. Problems in each chapter should be worked as they occur in the chapter. Exercises at the end of the chapters are designed to review and clinch ideas. Answers are found at the end of the book. A glossary of terms is provided for easy reference. Thirty chapters are included that consider various aspects of mathematics related to the K-6 curriculum. Emphasized are four strands: (1) Number Systems; (2) Geometry; (3) Measurement; and (4) Applications and Models. (RH)

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SCHOOL
MATHEMATICS
STUDY GROUP

STUDIES IN MATHEMATICS

VOLUME IX

A Brief Course in Mathematics For Elementary School Teachers

(revised edition)

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STUDIES IN MATHEMATICS

Volume IX

A BRIEF COURSE IN MATHEMATICS FOR ELEMENTARY SCHOOL TEACHERS

(revised edition)

Prepared by:

Max S. Bell, University of Chicago

William G. Chinn, San Francisco Unified School District,
San Francisco, California

Mary McDermott, Mount Diablo Unified School District, Concord, California

Richard S. Pieters, Phillips Academy, Andover, Massachusetts

Margaret Willerding, San Diego State College, San Diego, California

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PREFACE

For many years elementary school mathematics has been a study of numbers and their properties taught in terms of techniques and manipulations.

In recent years a number of people and organizations have worked to change this pattern. As a result there are now being advocated a number of "new" programs in elementary school mathematics. These programs have in common an increased emphasis on the structural aspects of mathematics and an attempt to show the "why" of arithmetical computation.

Now it is certainly fair for you to ask what all this fuss about "new programs" and "modern mathematics" involves and why you should concern yourself with such matters. An answer that is too obvious, but true nevertheless, is that mathematics as well as many other aspects of the world we live in is changing at a rapid rate. Less well known is the fact that in the past few decades the uses and applications of mathematics have also changed enormously, and, therefore, some sound knowledge of mathematics is becoming a prerequisite for fruitful work in an ever increasing number of occupations. This knowledge must include why mathematical processes work as well as how they work.

The new approach to mathematics can be greatly helped by using the new and better ways which have been developed in recent years of presenting knowledge in many fields to children. It is not enough for today's children to learn mathematics by rote memorization. Children now in elementary school will face problems we cannot predict. These problems will be solved not by knowledge of mathematical facts alone, but by knowledge of mathematical methods of attacking problems. New and as yet unknown questions may involve, and in fact require, new and as yet unknown mathematics for their solution. Naturally, we cannot teach this, but we must teach methods of mathematical thinking as well as the basic content of mathematics if we are to fulfill our responsibility to the youngsters in our charge.

In this period of transition it has been difficult for a teacher even to discover what mathematics is needed in order to teach the modern approach to mathematics in the elementary school, let alone to have a chance to learn it. We hope to point out to you some of these aspects of mathematics and to provide you with enough experience in them so that you can work better with children regardless of the particular program you are teaching. You will find most new programs have much in common. The common element is an attempt to lead students to understand principles rather than merely to present to them rules for memorization. It is important to note that a large part of what is presented here is background material for you as a teacher and is not intended to be transmitted to your students per se.

We know that you as teachers have some knowledge of mathematics. What we attempt to do in this course is to strengthen your understanding of mathematics and its uses. Hence, by the time you finish the course we hope you will have a better comprehension of mathematics; not merely of how to compute with numbers but of what numbers are and why we can work with them as we do. As you study this course, read the text and do the exercises you will discover that you are increasing your understanding of some of the basic notions underlying the mathematics that you are teaching.

The upgrading of the mathematics program involves new teaching methods, new ways of looking at the subject and new understandings of underlying principles as well as some new mathematics. At first you may find that this new way is not easier to teach but you will also find that the gratifying response of children to any program which gives the conceptual aspect its proper attention makes teaching and learning more exciting and more interesting.

Overview

The basic topics in the K-6 curriculum are typically Number Systems, including their properties and operations; Geometry; Measurement; and Applications. Figure P-1 shows briefly the main strands that will be dealt with in developing these topics from Kindergarten through Grade Eight.

	K		6
Numbers and Operations (Place Value)	Whole Numbers	Fractions, Rational Numbers (Positive and Negative)	Real Numbers
Geometry	Simple Figures	Point, Line, Plane Relations	Congruence, Similarity
Measurement	Lengths	Area, Standard Units	Volume
Applications and Models	Sets of Physical Objects	Problem Solving, Number Sentences	

Figure P-1

We may think of mathematics in three ways. One way is as an exercise in the manipulation of symbols. This way stresses the computational skills which are still important. Children must acquire computational skill and practice is necessary to develop this facility.

However, another way to think of mathematics is as an abstract system; children should understand the unifying, structural and organizational ideas which establish a mathematical system. These are the conceptual ideas which they will rely on for their understanding of each new concept in their mathematical learning.

Since the inspiration for mathematics comes from the physical world, applications are also essential. In fact, there must be a blending and a balance of all three aspects, the conceptual, the computational and the applications in the curriculum. It is to help you achieve this balance that is the aim and purpose of this course.

INTRODUCTION

In May, 1962, a conference of representives of the Mathematical Association of America, MAA, the National Council of Teachers of Mathematics, NCTM, and the School Mathematics Study Group, SMSG, was held. The purpose of the conference was to consider what further uses could be made of motion pictures and television to improve instruction in mathematics at all levels. One of the results of this conference was the suggestion that the SMSG prepare a course with mathematical content which could be used in the in-service training of elementary school teachers. The idea was to help them prepare to teach any one of the new curricula being suggested by various groups. It was also hoped that such a course might help any teacher understand better and therefore teach better any mathematics curriculum, old or new.

The SMSG accepted this suggestion. Its Panel on Elementary School Teacher Training prepared an outline of such a course. In the summer of 1962 there was prepared a preliminary version of a text to accompany the films when they were made. During the year 1962-63, thirty half-hour films were made with Professor Stewart Moredock as lecturer, using the outline and preliminary text as a guide line. In the summer of 1963, the text presented here was written bearing in mind the films as made and the many suggestions and comments received on the preliminary edition. This text is designed primarily to be used with the set of films mentioned before, providing further reading material as well as problems and exercises to help fix the ideas in mind.

In addition, however, it is felt that this material may be used independently of the films by teachers in elementary schools who wish to improve their knowledge of mathematics. It has been written with the idea that many of those who study it will not have consultants or professors at hand to answer questions. Hence, many details are included which may seem obvious to some but which, it is hoped, will be helpful to others.

Mathematics should always be studied with pencil in hand and lots of paper at the elbow. Problems in each chapter should be worked as they occur in that chapter as they are part of the development. Solutions for such problems may be checked immediately with those provided at the end of the chapter. Exercises at the end of the various chapters may serve to review and clinch the ideas presented therein. Answers for these will be found at the end of the book.

A glossary of terms which may be new and unfamiliar is provided for easy reference.

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Chapter 1

PRE-NUMBER IDEAS

Introduction to Sets

We cannot determine exactly the time when man began to use numbers. Surely in the very early stages of man's development even the wisest men knew very little about the numbers we use today because there was no need to do so. Presumably, food and shelter were obtained from what was available at the time and in the immediate environment. When the first crude forms of society developed, the necessity for keeping records of possessions became important and this involved some use of number. The basic ideas that underlie these first attempts at keeping records may seem very simple yet these are the ideas upon which our mathematical structure is built. It is interesting to note that primitive man's first attempts to solve situations concerning numbers correspond rather closely to the way young children think about number situations long before they have learned to count or to use numbers abstractly. When primitive man makes marks on the ground to keep account of his flock and then pairs each mark on the ground with each of his animals, he is going through essentially the same process that young children experience when they go to the cookie jar to get one cookie for each of their friends. In the first case the set of marks on the ground is matched with the set of animals. In the second case the set of cookies is matched with the set of children.

Sets

We think of numbers as abstract ideas about things. In fact numbers are abstractions and concepts derived from collections or sets of things. The concept of set is fundamental for communicating ideas in mathematics just as it is in everyday language. We speak of herds, flocks, committees, armies, teams, groups, etc. All of these terms may be replaced by the word set. A set is a collection of things and the things in the set are called the elements or members of the set. It is important to distinguish between the set and the elements within the set. If these are physical objects it is easy to construct the set by grouping the members and putting them in a container; then we know what is a member of the set and what is not a member. Another way of identifying the set is by listing its members.

Suppose we have a set such as a set of toys

A = {doll, ball, sailboat, airplane}.

For convenience the members of the set or the words that represent the members of the set are enclosed within braces. The set itself is indicated by the letter A. In set A all members have the common property of being toys. However, there need be no relation among the various members of a set other than being members of the set; nor is the order of listing members important.

B = {elephant, the color red, telephone, spaceship}

B = {the color red, spaceship, elephant, telephone}

In B there is no relation among the members of the set other than that they all belong to B. It is a set because its members are specified as being members of the set.

A set may have many members; it may have a single member; it may have no member at all. If a set has no member, it is named the empty set. Two examples of the empty set are

E = {mail carried by the pony express in 1963}

F = {jet planes that existed in 1963 B.C.}

The convention of using braces for sets is also used for the empty set. It is designated { }. The empty space between the braces indicates that there is no member of the empty set. Any example of the empty set has the same members as any other example of the empty set because none of them has any member. This is why we say the empty set; there is only one such set.

Sets are definite things. The elements of a set may be concrete objects such as an elephant or an abstract idea such as "the color red." Once the set has been given to us we can discover many ideas about it and about sets like it. One of these ideas about sets is the concept of number. In the next chapter we will explain and develop this concept from our intuitive knowledge of the properties of sets.

Summary

The notion of set is discussed as a key pre-number idea. By so doing we have the beginnings of a way by which we will be able to connect numbers to sets of physical objects. This connection is important because it enables arithmetic to be applied to the physical world and it is through working with sets that we effectively teach numbers to children.

Problems*

Pretend that you know nothing about number and you don't know how to count. How would you solve the following problems?

1. You have a handful of dimes that you wish to share with a friend. How can you make sure that you and your friend share equally?
2. Suppose you are interested in finding out whether there are more girls than boys in a school auditorium. What would you do?
3. There are two pastures of goats. One pasture is separated from the other by a very swift river. The two primitive tribesmen who own each of these pastures want to know which has more goats. They have a raft which will transport one of the tribesmen but will not transport the goats. There are pebbles scattered around both pastures. What might they do?
4. You own a set of tools that is so large that you cannot remember all that is in the set. How could you devise a scheme by which you can keep track of each tool?
5. You have as many textbooks as there are desks in your school room. The principal of the school notifies you that there will be as many children in your class as there are desks. What conclusions can you make concerning textbooks, desks and children?
6. Given two sets. How can you determine whether one set has as many members as the other set or whether it has more or less members than the other set?
7. You have a friend who speaks no English and you do not speak his language. How can you convey to him the idea of the number five using sets? How can you convey to him the idea of a triangle using sets?

Solutions for Problems

1. Pair each dime that you give to your friend with each dime that you keep for yourself.
2. Line up boys on one side of the auditorium and girls on the other side. Pair off one boy to one girl.
3. Each tribesman collects one pebble for each of his goats. One tribesman crosses the river with his set of pebbles. The two tribesmen pair their pebbles one by one. If there are more pebbles in one man's set than in the other man's set, he has the greater number of goats.

* Solutions for problems in this chapter are on this page.

4. Draw an outline of each tool on the board where they hang. If the set of tools doesn't match the set of outlines, there are missing tools.

5. You can put a book on each desk and seat a child at each desk. This pairs the books and the desks, and also pairs the children with the desks. Thus there is one book for each child and one child for each book. We have the same number of books as children.

6. By pairing the elements of one set with those of the other. If the pairing comes out even, each set has as many members as the other. Otherwise the set which has some elements left unpaired has more members than the other.

7. You may point to many sets of objects each having five members. You may display many representations and various models of triangles and by tracing with your finger the outline of each, convey the idea of triangle.

Chapter 2

WHOLE NUMBERS

Introduction

In Chapter 1 the idea of a set was introduced and the statement made that out of some of the basic properties of sets the idea of number would be developed. The problems at the end of the chapter were carefully designed to introduce some of these properties which will be more carefully developed in this and the following chapters. Many times in this book we will talk about physical models to help clarify ideas. It will help you both now and later to actually make such physical models and work with them as you read. Different kinds of buttons, coins, checkers, etc., would be excellent as members of sets which you could use to illustrate the various situations we shall talk about.

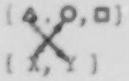
Description of a Set

The first basic idea leading to the notion of number is the idea of a set itself which we began to look at in Chapter 1. An important part of the idea of a set is that a set is completely specified when its members are specified. Thus the set {Mary, Bill, Max, Dick} is the same set as the set {Dick, Mary, Max, Bill}. The same set may well have many different descriptions. In fact, the set named before may very well be the same as the set of students now working at the chalkboard. Thus a set may be determined by naming its members (without regard to their order) or describing them by some property they have in common. Remember that if a set has no members or there are no objects which possess the specified property, the set is called the empty set. Thus the set of purple cows is { }.

Pairing the Members of Two Sets

A second basic idea leading towards number involves two sets. Suppose that we have two specific sets of objects A and B where A = {△, ○, □} and B = {X, Y}. We can think of pairing the individual members of A with those of B. To carry out this operation, we choose one member, in any way we wish, from the first set and at the same time one member from the second set. We put these two objects aside. Next we repeat the process, choosing one of the remaining members of the first set and one of the remaining members of the second. We put these aside, and then continue. We keep going until we run out of members of one of the sets (or perhaps both at the same time). For example, we start a one-to-one pairing of the members of A and B by choosing the ○ from A and the X from B. We put them aside and for the second step we choose the △ from A and the Y from B. Now we are finished since we have used all the members from B even though the □ is left in A.

Another way of picturing this is to connect the members of the first set with the members of the second set that they are paired with. The example above can be pictured in this way:



When we pair two sets, there are only three possible outcomes. In the first place, we might run out of members of the two sets at the same time. In this case, we say that the sets match. For example, these two sets match: {△, ○} and {X, Y}, but these two do not: {△, ○, □} and {X, Y}.

In the second place, we might use up all the elements of the second set before running out of members of the first. In this case, we say that the first set is more than the second.

For example: {△, ○, □} is more than {X, Y}. A is more than B.

Finally, we might use up all the members of the first set before those of the second. In this case we say that the first set is less than the second. For example, {△, ○, □} is less than {W, X, Y, Z}.

The most important fact about the operation of pairing is that the outcome does not depend on the order in which we pick out the members. Thus, if we pair two sets and discover that they match, then we can be sure that if we shuffle the members of the first set and also shuffle the members of the second set and then repeat the operation, the outcome will be the same. They will still match. Whether or not two sets match depends only on the sets and not on the way the members of the sets are arranged.

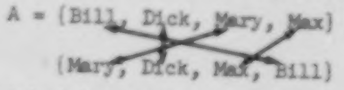
It is this one-to-one pairing of the members of two sets which is the method you should have used to answer the problems posed in Chapter 1. If you have the boys and girls in the auditorium pair up, you can quickly tell if there are as many girls as boys. If you paint a picture of each tool you own on a peg board and hang each tool in front of its picture, it is obvious if one is missing.

Matching Sets

If there is a one-to-one pairing of all the members of one set with all the members of another set, we have a matching of the two sets and a one-to-one correspondence of their members.

There are some obvious but important properties of this relation between sets which we call matching.

1. It is always true that a set matches itself. Thus if



$$A = \{\text{Bill, Dick, Mary, Max}\}$$

then the set $\{\text{Mary, Dick, Max, Bill}\}$ is the same set A and the sets can be matched by pairing each child in the first listing with himself in the second.

2. If A and B are any two sets and if A matches B, then B matches A.

This is true since the one-to-one pairing of each member of A with a member of B can be considered equally well as a one-to-one pairing of each member of B with a member of A.

3. If A, B and C are any three sets, and if A matches B and B matches C, then A matches C.

Thus in the problem posed in Chapter 1 about the school children, textbooks and desks, we can put a book on each desk, since we know these sets match. Also, we can seat each child at a desk and know that there are no empty desks since these sets match. But now we have a one-to-one pairing of books with children and so we know the set of books matches the set of children.

The three properties listed above enable us to talk about a whole collection of matched sets since if D matches any one of the three matched sets A, B and C, it will match the other two and so on for any other set which matches A, B, C or D. We will return to this collection of matched sets shortly.

The More Than Relation

When we paired off the members of A with those of B, it might have happened that we ran out of members of B before we had used up all the members of A or vice versa. In the first case we said that A was more than B. Thus the set {△, ○, □} was more than the set {X, Y}. We also say that {X, Y} is less than {△, ○, □}. The set C in Figure 2-1 is more than the set D since in the one-to-one pairing of members illustrated we run out of members of D before we use up all the members of C.

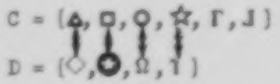
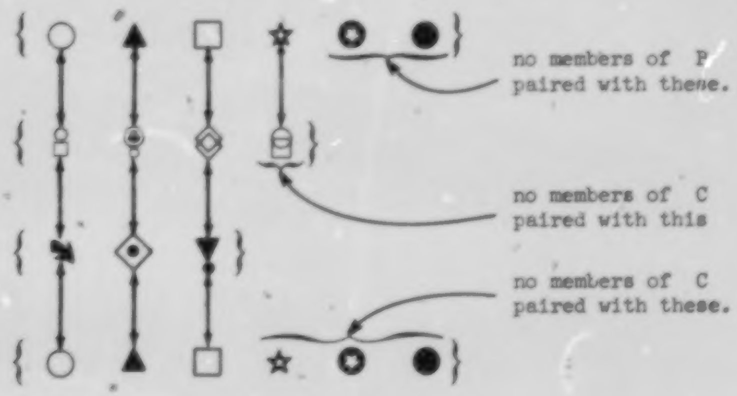


Figure 2-1. Set C which is more than set D.

Just as there were some important properties of the matching relationship between sets, so there are some important properties of the "more than" relationship.

1. If A, B and C are any three sets and if A is more than B and B is more than C, then A is more than C.

This follows most clearly from consideration of Figure 2-2.


no members of B paired with these.

no members of C paired with this

no members of C paired with these.

Figure 2-2. If A is more than B and B is more than C then A is more than C.

2. If A and B are any two sets such that A is more than B and if C is any set which matches A and D is any set that matches B, then C is more than D.

This also follows from a figure such as Figure 2-3.

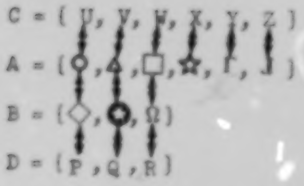


Figure 2-3. Another property of "more than."

We can pair each member of D with a member of C by tracing through the pairings from a member of D to a member of B to a member of A to a member of C. But when all the members of D are thus paired, there are still some members of C, such as Z, which are not paired with any member of D. Thus C is more than D.

Finally we mention again, for emphasis, a property mentioned before. That is:

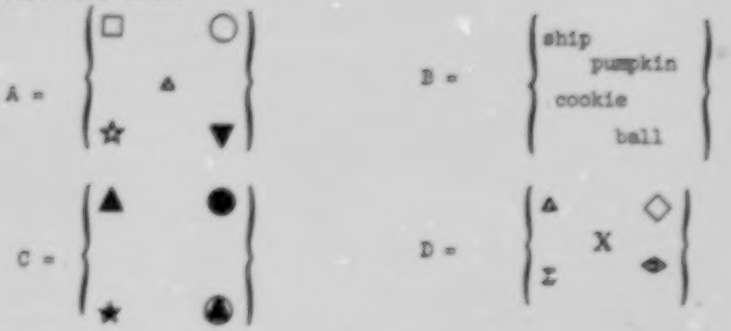
> If A and B are any two sets, then just one of the following occurs: A is more than B; A matches B; or A is less than B.

We can see that this must be true by thinking about the pairing of members of A and B. Either we run out of members of B before we do those of A or we have just enough or there are members of B left over when we run out of those of A. These situations correspond exactly to the three cases listed above.

It should be mentioned here that the relationship "less than" has properties corresponding exactly to those of "more than" since to say "B is less than A" is to say exactly the same thing as "A is more than B."

Problems*

1. Here are some sets:



   a. Which pairs of sets match? Draw arrows to show the matching.

   b. Which sets are less than A? Draw arrows to show the pairing, and show the members of A left over.

2. Give an example of sets A, B and C such that A is less than B and B is less than C. Draw arrows to show that A is less than C.

3. Give examples of sets A, B, P and Q with A less than B, P matching A and Q matching B. Draw arrows to show that P is less than Q.

4. Which of the following sets are empty sets?

   a. The set of girls ten feet tall.

   b. The set of boys five years old.

   c. The set of women who have been President of the U.S.A.

   d. The set of bald headed men.

   e. The set of children with brown hair.

Properties Common to Some Sets

Now what do all these considerations of sets, pairing elements of sets, sets matching or being more than or less than other sets, and collections of matching sets have to do with numbers? Numbers and their properties are ideas associated with sets and properties of sets, but how can we get hold of ideas just by looking at sets? Consider how you might try to get across the idea of a triangle to someone who speaks no English. You could show him a set of objects of triangular shape. Out of a whole mass of different shaped objects you would pick up a triangular one and put it in the set, but a circular one you would reject. After seeing this kind of selection made several times, your friend would probably get the idea. Similarly, if you wanted to explain the word "blue" to someone, you could put into one set a blue sweater, a blue coat and a blue hat. At this point he might think "blue" meant something to wear, but if you added a blue flower, a book with a blue cover and a blue electric light bulb, he might get the idea. If not, we could continue adding blue objects to the set, rejecting red and green ones, until he could make the correct decision himself thus indicating that he has the idea.

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* Solutions for problems in this chapter are on page 18.

A Collection of Matched Sets

So it is with the sets we have been talking about. What is the property we want to acquire out of our consideration of sets of objects? This time it is not the color or shape of an object which determines whether it goes into our collection, it is whether a certain set under consideration matches a given set. Thus these sets {○, ▲, □, ☆, Ω} and {P, Q, R, S, T} share a common property; they can be matched with each other. There are many other sets with the same common property, i.e., sets such as {•, ◇, ✪, Γ, J} or {boy, cow, dog, cat, pig} which can also be matched with each of them. The property shared by all the sets which can be matched with these is the elusive idea that we are after. It is called the number property of these sets. For these particular sets, we choose to call the number property "five" and to write "5." All the sets which match the set {○, ▲, □} share a different number property which we call "three" and write as "3." Of course, many other sets of objects share this number property. The set {Joe, Jane, John} has the number property 3; the set {boat, house, car, garage, bank} has the number property 5. In order to write this easily, we use $N(A)$ as an abbreviation for the phrase "the number property of A." Since this number property is a property shared by all the sets which match A we can say that if A and B are sets which match, then they have the same number property and we can write $N(A) = N(B)$.

Number

It is important to realize that the number property associated with {▲,○,□} is something which has many different names. A Roman knew this, he called it tres and wrote III; a Chinese knows it, he calls it sahn and writes 三 ; a Frenchman and a German know it, they call it trois and drei, but write it 3 as we do. There are many names by which this number property may be called, they are the numerals, III, 三 , 3, etc. But it is the number property shared by all the sets which match {▲,○,□} which is the number itself. It isn't the name, the numeral, which is important; it is the recognition that

> a number is the common property shared by a collection of matched sets

which is the important idea we must get. Such numbers are called whole numbers and we see that they are connected fundamentally with sets of objects. The properties of these whole numbers will therefore follow naturally from the properties of sets which were so carefully listed a few paragraphs back.

With each collection of matched sets we associate a certain number: with the sets which match {X, Y}, the number 2; with those which match {P, Q, R, S}, the number 4; with those which match {□}, the number 1; with the empty set { }, the number 0; and so on. We say the empty set because there is only one. The idea of zero as a number is a difficult idea for many people. As a matter of fact, it was historically late in being accepted. But it is extremely useful, in fact almost vital for us.

Problems

5. Name the number property associated with the collection of sets which match the following sets.

   a. {□,▲}
   b. {R, S, T, U, V}
   c. { }
   d. {□}
   e. {r,s,t,u,v,w,x,y,z,p,q}
   f. {living members of the U.S. Senate}
   g. {the people in your immediate family}
   h. {the things in your handbag or pocket}

6. Specify sets for which each number given would be the number property associated.

   a. 4
   b. 5
   c. 3
   d. 0
   e. 9
   f. 1

Properties of Number

When two sets are compared by a one-to-one pairing of their members, we find that A may match B, that A may be more than B, or that A may be less than B and these are the only things that can happen. Corresponding to these three situations, we can say in the first case N(A) and N(B) are equal, $N(A) = N(B)$; in the second case N(A) is greater than N(B), $N(A) > N(B)$; and in the third case N(A) is smaller than N(B), $N(A) < N(B)$. Note the symbols $>$ and $<$ to express the relationship "greater than" and "smaller than" between numbers. How are we to decide between two numbers, say 4 and 6, as to which one is the greater? We go right back to a set out of the collection associated with 4 and a set out of the collection associated with 6. We know we can decide by a one-to-one pairing of members of these sets that a set of 6 is more than a set of 4 and therefore we say that $6 > 4$ or $4 < 6$.

Ordering the Whole Numbers

We can compare any two whole numbers in this way. Therefore we can now take any set of whole numbers, such as {6, 3, 2, 0, 4, 1, 5}, compare them in this fashion, and then order them by putting them in a row with the smaller ones to the left. Thus we soon find by comparing their sets, that $2 < 4$ and we already know $4 < 6$. So we order them 2, 4, 6. What about 3? Comparing sets of 3 with sets of 2 we find $2 < 3$, so 3 goes to the right of 2 but where in relation to 4 and 6? Comparing a set of 3 with a set of 4 we find $3 < 4$. Now we have 2, 3, 4, 6. In like manner we consider the numbers 5, 0, 1 and discover that the correct ordering is 0, 1, 2, 3, 4, 5, 6.

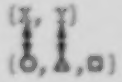
Problem

7. Put each of the following sets of numbers in order.

a. {3, 1, 2}	d. {4, 3, 2, 1, 0}
b. {6, 0, 4, 5}	e. {5, 1, 4, 3, 2}
c. {5, 7, 3, 1, 9}	f. {1, 4, 3, 7}

Successive Numbers

If we look at sets of 2 and 3 we see that in the pairing



the set of members left over after all the possible pairings is a set, {□}, with exactly one member. Any set, therefore, which is more than a set of 2 must either match a set of 3 or be more than a set of 3. This means that there is no whole number which is greater than 2 and also smaller than 3. We say that 3 is one more than 2.

Any number such as 6 is the number property of a set of matched sets. Take one such set. It is always possible to put another object in this set. Thus we can put ⭘ into the set A = {▲, ○, □, ☆, Ω, Γ} and get the set B = {▲, ○, □, ⭘, ☆, Ω, Γ}. This is one of a collection of matched sets and to this collection is associated a number N(B) which is one more than 6. Of course, this is the number we call 7. But the important thing to realize is that for any number we can always go through the same procedure and find another number which is one more than it. How can we possibly name all these numbers? This is a very real question to which many different answers have been given. Chapter 3 will be devoted to exploring the different answers and deciding which one is best for us.

Counting

The ordering of the whole numbers which we have achieved, 0, 1, 2, 3, ..., is, of course, the basis of what we call counting. For this purpose we drop the 0 and consider the ordered set of the so called "counting numbers," 1, 2, 3, 4, ... There is a very remarkable fact about a set of these numbers which in some ways seems so obvious that it is hard to appreciate. If we take any set of these counting numbers starting with 1 and going up say to n, we have a set of objects. We might well ask: To what collection of matched sets does this set belong? In other words: What is its number property? Or in still other words: How many objects are in this set? Now the remarkable property of this set is that the last number n in the ordered set we selected is exactly the answer to this question. This is why we can use such a set to count the members of a given arbitrary set A. If we want to count the members of such a set, we pair them off one by one with the ordered set of counting numbers until we run out of members of A. Thus, to count the set {▲, ○, □, ☆} we might pair them off in this fashion: □ ⟷ 1, ○ ⟷ 2, ☆ ⟷ 3, ▲ ⟷ 4. The last number which we use in this pairing is the number which "counts" the set and names the number property of the set. This is just because we have achieved a matching of the members of this set with the members of a set of the ordered numbers and we know the last number, 4, of this set tells us the number of members in the set. Exactly the same procedure will work for an arbitrary set A and the pairing of its members with the counting numbers 1, 2, 3, ..., n tells us that A has _n_ members.

Number Sentence

When we talk about numbers and say for example that six is greater than four or write $6 > 4$ we are using numbers in a sentence or writing a "number sentence." Frequently, we will want to talk about numbers or write a number sentence about two numbers without specifying which numbers we mean. For example, we may want to speak about the number property of sets A and B. We have written this as $N(A)$ or $N(B)$. We also use letters such as _a_ and _b_ for this purpose, i.e., to represent numbers. Thus if A is more than B we write $N(A) > N(B)$ or $a > b$. This also is a number sentence. To say $a = b$ is to say that A and B are both sets from the same collection of matched sets and therefore have the same number property. Number sentences are going to be very useful to us and will be studied carefully in later chapters.

A Number Line

Another very useful device in our study of numbers is to think of them in relation to a "number line." We simply draw a line as in Figure 2-4a with arrows on both ends indicating that it can go on and on if we want it to.

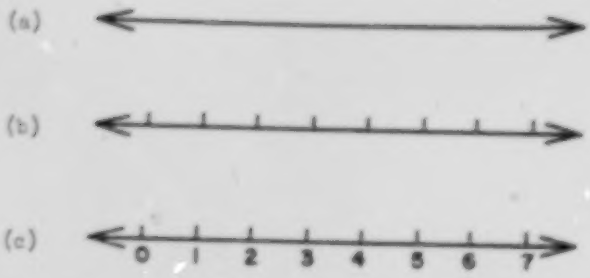


Figure 2-4. A number line.

Then we mark on it as in Figure 2-4b a number of equally spaced marks or dots. And then starting at the left we label the dots with the whole numbers in order, 0, 1, 2, 3, 4, 5, 6, 7, ... The arrows on the number line indicate we can extend it if we want to; the three dots ... at the end of our list of numbers indicate that we can extend it if we want to. Now the order of the whole numbers shows up clearly by the position of the labelled dots. The dot labelled 7 is to the right of the dot labelled 5. Correspondingly $7 > 5$. Of course, this is so since we labelled the dots in order, but sometimes pictures and diagrams help us to see and retain ideas more easily. The number line is going to be extremely useful to us again and again in our future work.

Summary

Let us summarize briefly what we have done. We considered sets and their properties and from these ideas developed the idea of number. The properties of whole numbers correspond to the properties of sets. We list some of these in Figure 2-5.

Sets	Whole Numbers
The one-to-one pairing of members of A and B results in either	For any whole numbers either
A matches B	$N(A) = N(B)$ or $a = b$
A is more than B	$N(A) > N(B)$ or $a > b$
A is less than B	$N(A) < N(B)$ or $a < b$
If A is more than B	If $a > b$
and B is more than C	and $b > c$
then A is more than C.	then $a > c$.
If A is less than B	If $a < b$
and B is less than C	and $b < c$
then A is less than C.	then $a < c$.

Figure 2-5. Correspondence between numbers and sets.

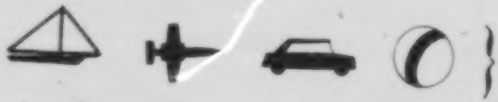
The whole numbers can be arranged in an order and this order used as the basis of counting.

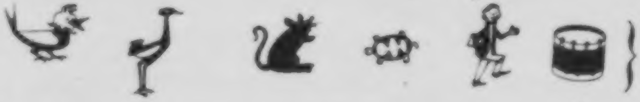
Numerals are names for numbers. The same number may have several different numerals. The problem of names for numbers, particularly large numbers, is the topic of Chapter 3.

Exercises - Chapter 2

1. Which sets below are of the same type?
   a. {a, b, c, d, e, f}
   b. {□, ○, ◉; △}
   c. {1, 2, 3, 4, 5, 6}
   d. { }
   e. {△, ◈, □, ●, ☆}

2. Match the elements in Set A with the elements in Set B and tell which set has more elements.

A = {  }

B = {  }

3. If the elements of set A match with the elements of set B and the elements of set B match with the elements of set C, what can you say about the elements of sets A and C?

4. What numbers are associated with the following sets?
   a. { }
   b. {▲, □}
   c. {▲, □, ●, △, ○}
   d. {∫}
   e. {○, ●, ◉, ⊖, ♁, ◬, ♇, ♉, ⚲, ⓨ}

5. Which of the following sets are empty sets?
   a. The set of girls fifty feet tall.
   b. The set of married men.
   c. The set of ugly women.
   d. The set of men from Mars.
   e. The set of children in your school.

6. Construct a set A and a set B and write a number sentence to show that A is less than B.

7. Write number sentences using symbols to show the number property of A and the number property of B.

8. Suppose you want to explain "wide" to someone who speaks no English and you do not speak his language. How would you go about conveying to him the idea of "wide"?

9. Write number sentences using appropriate symbols to show the relationship that exists among these sets.
   A = {△, ○, ⧅, Γ, ψ}
   B = {a, b, c, d}
   C = {p, q, r}

10. List several symbols for real objects.

11. What is the smallest whole number represented on the number line?

12. What can you say about every number represented by a point on the number line that lies to the right of a given point?

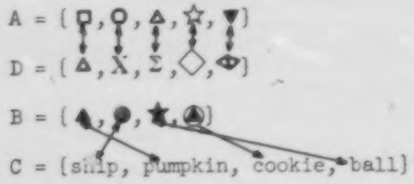
13. If one number is greater than another, what do you know about their places on the number line?

14. What do the arrows on either end of the number line indicate?

Solutions for Problems

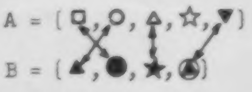
1. a. A matches D,

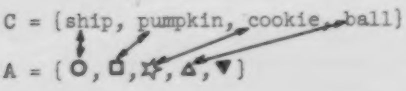
B matches C



Many other pairings are possible.

b. C and B are both less than A.



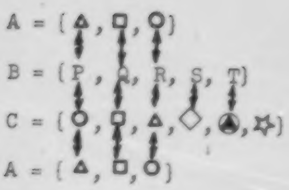


☆ is left over.

▼ is left over.

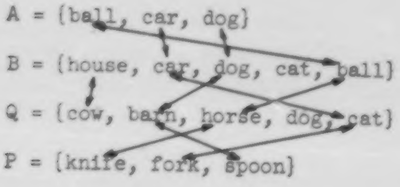
Many other pairings are possible but in each case some member of A will be left over.

2.



◇, ●, ☆ are left over.

3. P = {knife, fork, spoon}


A = {ball, car, dog}
B = {house, car, dog, cat, ball}
Q = {cow, barn, horse, dog, cat}
P = {knife, fork, spoon}

cow, dog are left over.

4. a and c are empty sets as of 1963.

5. a. 2
   b. 5
   c. 0
   d. 1
   e. 11
   f. 100 in August, 1963.
   g., h. } only you can give these.

6. a. {boy, girl, man, woman}
   b. {hammer, saw, drill, nail, screw}
   c. {X, Y, Z}
   d. { }
   e. {9, 6, 5, 3, 1, 2, 4, 8, 7}
   f. {P}

7. a. {1, 2, 3}
   b. {0, 4, 5, 6}
   c. {1, 3, 5, 7, 9}
   d. {0, 1, 2, 3, 4}
   e. {1, 2, 3, 4, 5}
   f. {1, 3, 4, 7}

Chapter 3

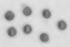
NAMES FOR NUMBERS

Introduction

In the preceding chapter we studied briefly the idea of a set and its elements. We found that some sets had the property that their elements could be matched with each other in a one-to-one correspondence. From such sets we extracted the idea of number as a property of a set of matching sets. This was called the number property of a set. We found that some sets had "more" elements than others and out of this came the idea of the "order" of numbers and the possibility of using numbers to count the elements of a set. Numbers can be represented by points on a number line and the order of numbers corresponds to a sequence of the points assigned.

The Beginnings of Numeration

Now the problem must be faced as to how the numbers are to be named and what symbols can be used for writing them. This is a problem which has received attention over many years from the early days of the cave man down to the present era when the advent of high speed computing machines has forced us to look anew at the question.

In primitive times, men were probably aware of simple numbers in counting, as in counting "one deer" or "two arrows." Primitive people also learned to use numbers to keep records. Sometimes they tied knots in a rope, or used a pile of pebbles, or cut marks in a stick to represent the number of objects counted. A boy counting sheep might have  pebbles, or he might make notches in a stick, as . One pebble, or one mark in a stick would represent a single sheep. The same kind of a record might be made by tally marks as are used even to this day in, for instance, counting votes in a class election, 卌 || . When people began to make marks for numbers, by making scratches on a stone or in the dirt, or by cutting notches in a stick, they were writing the first numerals. Numerals are symbols for numbers. Thus the numeral "7" is a symbol for the number seven as are the marks above. Numeration is the study of how symbols are written to represent numbers.

Distinctions Between Words

In working in any field in mathematics, the distinction between different words is usually very important. Thus the word "numeration" used here is not the same word as "enumeration" with which you may be more familiar. An enumeration of your class is a counting off of the members of your class.

Principles of numeration cannot be developed effectively if confusion exists regarding the terms number and numeral. These are not synonymous. A number is a concept, an abstraction. A numeral is a symbol, a name for a number. A numeration system is a numeral system, not a number system, for naming numbers.

Admittedly there are times when making the distinction between such words as "number" and "numeral" becomes somewhat cumbersome. However, an attempt should be made in treating elementary arithmetic to use such terms as "number," "numeral" and "numeration" with precise mathematical meaning.

This may be an appropriate time to comment on the correct use of the equal sign "=". For example, when we write

$$5 + 2 = 8 - 1$$

we are asserting that the symbols "5 + 2" and "8 - 1" are each names for the same thing—the number 7. In general, when we write

$$A = B$$

we do not mean that the letters or symbols "A" and "B" are the same. They very evidently are not. What we do mean is that the letters "A" and "B" are each being used as names for the same thing. That is, the equality

$$A = B$$

asserts precisely that the thing being named by the symbol "A" is identical with the thing being named by the symbol "B". The equals sign always should be used only in this sense.

It is imperative to recognize that a number may have several names. The symbols VII, 7, 8 - 1, 2 + 2 + 3 are all names for the same number, seven.

Numeration Systems

It is of interest to look at some different systems of writing numerals and to contrast them with our own Hindu-Arabic System to see the advantages inherent in it.

The earliest systems started out with tally marks, i.e., | for one, || for two, ||| for three, etc. You can see that we will run into difficulty quickly under this system, particularly when we wish to write large numbers. If we examine different numeration systems, such as the Egyptian, the Chinese and the Roman, we find that tally marks, letters and various forms of special symbols are always used for small numbers. It is when the necessity for naming large numbers arises that the advantages of certain systems of numeration over others become apparent.

To avoid using too many symbols, a process of grouping evolved very early in the history of man's civilization. Just what grouping means and how it helps will be explained in this chapter. Some numeration systems grouped by twenties, some by twelves, some by twos, but almost universally grouping was by tens. This may be because counting began by matching fingers and we have ten fingers on our two hands. Vestiges of other groupings in our language are the words "score," "dozen," "couple," etc.

Egyptian Numerals

One of the earlier systems of writing numerals of which there is any record is the Egyptian. Their hieroglyphic, or picture numerals, have been traced back as far as 3300 B.C. Thus about 5000 years ago Egyptians had developed a system with which they could express numbers up to millions. Egyptian symbols are shown in Figure 3-1.

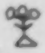
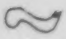
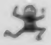
Our Symbol	Egyptian Symbol	Object Represented
1	/ or |	stroke or vertical staff
10	∩	heel bone
100		coiled rope or scroll
1,000		lotus flower
10,000		pointing finger
100,000		burbot fish (or polliwog)
1,000,000		astonished man

Figure 3-1. Egyptian symbols for numbers.

The Egyptian system combined a simple tally system with the idea of grouping by tens. Thus, one was represented by a single mark /, and

///
nine by /// ; note grouping by 3's for ease in reading. For ten, instead
/// ///
of writing ///, they considered all these possible tallies as being
///
/

combined into a single group and represented it by a new symbol ∩. Then instead of writing ten ∩'s for a hundred, they considered this as a single group and introduced again a new symbol 𓍢 for it, and so on. This is the fundamental idea of grouping which is an important advance in numeration systems.

Using the Egyptian symbols, we can write two hundred thirty-three as 𓍢𓍢 ∩∩∩|||. Note that in this system we need symbols for each of the various groups such as ones, tens and hundreds. Because of the tallying idea we find that nine hundred eighty-seven is awkward to write since the symbols must be repeated so often. Thus

𓍢𓍢𓍢∩∩∩|||
𓍢𓍢𓍢∩∩∩||| = 987.
𓍢𓍢𓍢∩∩|

The Egyptian system was an improvement over the cave man's system of tallies because it used these ideas:

1. A single symbol could be used to represent the number of objects in a group. For example, a heel bone represented a group of ten.
2. The system was based on groups of ten. Ten strokes made up a heel bone, ten heel bones made a scroll, etc.
3. Symbols were repeated to show sums of numbers. Thus the symbols 𓍢𓍢𓍢 mean 100 + 100 + 100 or 300.

Our numerals	4	11	23	20,200	1963
Egyptian numerals	||||	∩|	∩∩|||	𓂭𓂭𓍢𓍢	𓆼 𓍢𓍢𓍢∩∩∩ 𓍢𓍢𓍢∩∩∩||| 𓍢𓍢𓍢

Figure 3-2. A table showing how Egyptians wrote certain numbers.

The prime disadvantage of the Egyptian system is that they used no symbol for the number zero and so their numeration system, while using the same base ten as we ordinarily do, could not use the idea of place value.

Problems*

1. Write numerals for the following numbers in tallies and in Egyptian symbols.

   a. 9 c. 8

   b. 12 d. 19

2. Write in Egyptian symbols.

   a. 232 c. 2354

   b. 1111 d. 6002

Other Numeration Systems

In many ways the Roman numeration system is the same as the Egyptian except that the Romans used different symbols and added symbols for five and fifty so that there did not have to be quite so many repetitions in writing certain numerals. The common Roman symbols are shown in Figure 3-3.

Our numeral	1	5	10	50	100	500	1000
Roman numeral	I	V	X	L	C	D	M

Figure 3-3. Roman symbols for numbers.

Historians believe that the Roman numerals probably came from pictures of fingers like this, |, ||, ||| and ||||. They then used a hand for five. Gradually they omitted some of the marks and wrote V. Two fives put together gave X the symbol for ten. The other symbols were letters of the alphabet. In this system 987 would be written as DCCCCLXXXVII. In later Roman times, they adopted a subtractive idea where if an X were written before C, the numeral XC would stand for ninety, whereas CX would stand for one hundred ten. This is the system we see used to this day for inscriptions of dates on buildings, etc. In this fashion 987 would be written as CMLXXXVII.

The Chinese developed a numeration system which avoids the tedious repetition of symbols as tallies. They had symbols for each digit from one to nine and symbols for ten, hundred, thousand, etc. To write five hundred they wrote the symbol for five and the symbol for hundred. To illustrate this we will use our own symbols for digits and the Egyptian symbols for ten and hundred. Thus, three hundred would be written as 3𓍢 instead of 𓍢𓍢𓍢 , and 987 would appear as 9𓍢 8∩ 7. Notice how fewer and fewer symbols are needed as the system gets better and better. What is still missing? Curiously enough it is the symbol for the number of the empty set.

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*Solutions for problems in this chapter are on page 30.

Problems

3. Write in Roman numerals.

a. 232	c. 2594
b. 1111	d. 1963

4. Write in Chinese system using Arabic digits and Egyptian symbols as in the text.

a. 2594	c. 452
b. 1963	d. 6020

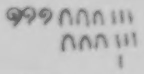
0 and the Decimal System

Further advances in numeration systems had to wait for the invention and general acceptance of the symbol 0 for the number zero. Systems up to this point always needed special symbols for groups of tens and hundreds. If the value of a symbol is going to depend, as it does in our system, on the position it occupies in the written numeral, we must have a method and a symbol to determine its position. In 987 the value of the 9 as nine hundred is determined by its being in the third position. But, if the 8 and the 7 were not there to hold the tens and units positions, we could not tell whether the 9 represented nine hundred, ninety, nine thousand or even just plain nine. In saying nine hundred seven we know that we have nine hundreds, but if we had to write the numeral without the symbol "0" we might try 97 or even 9 7 but we would have no way of being sure what the nine represents because we do not know what position it is supposed to be in. The symbol 0 enables us to write 907 and be sure the 9 stands for nine hundred since the 0 pushes it over into the third position. It is this use of 0 to establish the positional character of our numeration system that distinguishes it from all the earlier ones. It is this, combined with our practice of grouping by tens and hundreds, that gives us our decimal place value system. The word "decimal" comes from the Latin word decem which means ten. It is used to indicate that the basic grouping is by tens.

This system is said to have the <u>base</u> ten. After looking at our own system with base ten, we shall also look briefly at systems with different bases.

<u>Problem</u>

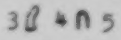
5. Write the following numbers in our own system.

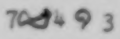
a. 

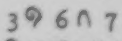
b. DXVII

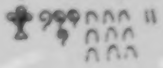
c. XCIX

d. MCDXCII

e. 

f. 

g. 

h. 

Let us now analyze our decimal place value system. We are going to group by tens and hundreds and are going to indicate groups by positional places in the numeral. First we need symbols for all the numbers up to ten and, as we saw, a special symbol for zero. These are the familiar digits, 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9, the so called Hindu-Arabic numerals. Written in the first position these digits indicate how many units there are in the number. Next to indicate the number of tens, we write the appropriate digit in the second position to the left, holding open the first position by writing a 0 there if another digit is not required. In the same way we write a digit in the third position to represent the number of hundreds required and so on in the fourth and fifth positions for thousands and ten thousands, etc. In this system a digit in any position in a numeral has a value ten times that which it would have if written one position to the right. Thus, the 7 in 7063 means seven thousand, while in 6752 it stands for seven hundred and in 1673 for seventy.

How would we write a decimal numeral to indicate the number of x's in the following set?

```
x  x  x  x  x  x   x  x  x  x  x  x  x  x
 x x x x x x x x x x x x x x x x x x
x   x   x   x   x   x   x   x   x   y
x x x x x x x x x x x x x x x x x x x x x x x x x x x
x x x x x x x x x x x x x x x x x x x
x  x  x  x  x  x  x  x  x  x  x  x  x  x x
```

Figure 3-4. A set of x's.

We group these by tens and hundreds in Figure 3-5.

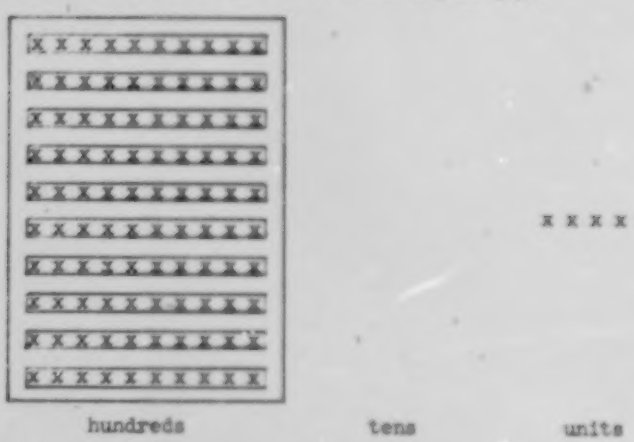


Figure 3-5.

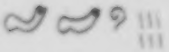
Analyzing this grouping we see that we have one group of hundreds, no group of tens and four units. So the numeral can be written as 104. Without the 0 to hold the second place or position open, how could we have written a symbol for this number?

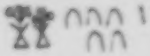
The decimal place value system adds the powerful idea of positional value to the method of grouping to give us a numeration system peculiarly adapted to ease of writing and ordinary computation. It is these facts and not the fact that the base of the system happens to be ten that make it so good. In fact, if we examine systems with different bases which still use the idea of grouping and positional value, we shall gain greater understanding of our own system. This we shall do in the next chapter.

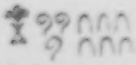
Exercises - Chapter 3

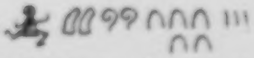
1. Which of the numerals below are names for the same number?
   a. III b. 111 c. 3 d. 7 - 4 e. 5 - 1
2. List six numerals that are symbols for five; for eight; for four.

3. Write decimal numerals for each of the following:

a. 

b. 

c. 

d. 

4. Explain the difference in meaning of the Roman II and the decimal notation 11.

5. From the list below write all the letters which are beside correct names for 467.

a. Four hundred sixty-seven
b. Forty-six and seven more
c. Forty-six tens and seven
d. Forty hundreds + sixty-seven ones
e. 300 + 160 + 7
f. Seven plus four hundred
g. 400 + 60 + 7
h. 300 + 150 + 17
i. 467 tens

6. Answer Yes or No.

a. 3,729 is 37 tens plus 29 ones.
b. Ten hundreds plus forty tens plus nine ones is the same as one thousand forty-nine.
c. 5,000 + 500 + 1 = 5,501.
d. 36 hundreds + 1 ten + 18 ones = 3,628.
e. 734 = 600 + 120 + 24.

7. Write the base ten numeral for each.

a. Five thousands + six hundreds + eight tens + three ones
b. 3 thousands + 8 hundreds + 16 tens + 5 ones
c. 6 thousands + 15 hundreds + 2 tens + 7 ones
d. 8 thousands + 4 hundreds + 14 tens + 16 ones
e. 9 thousands + 12 hundreds + 3 tens + 14 ones

Solutions for Problems

1.	Tallies	Egyptian		Tallies	Egyptian
a.	𝍸 ||||	||| ||| |||	c.	𝍸 |||	||| ||| ||
b.	𝍸 𝍸 ||	∩||	d.	𝍸 𝍸 𝍸 ||||	∩||| ||| |||

2. a. 𓍢𓍢∩∩∩||

b. 𓆼𓍢∩|

c. 𓆼𓆼𓍢𓍢𓍢∩∩∩|||
∩∩|

d. 𓆼𓆼𓆼 ||
𓆼𓆼𓆼

3. a. CCXXXII

b. MCXI

c. MMDXCIV

d. MCMLXIII

4. a. 2𓆼5𓍢9∩4

b. 1𓆼9𓍢6∩3

c. 4𓍢5∩2

d. 6𓆼2∩

5. a. 367

b. 517

c. 99

d. 1492

e. 30,045

f. 700,403

g. 367

h. 1492

Chapter 4

NUMERATION SYSTEMS

What is it that makes our system of numeration superior to those of ancient times? Essentially it is the idea of place value, which became easy to use only after the number zero and the numeral 0 were introduced. Some people feel that the use of ten as the base of our system is essential to its success, but probably the only reason we use ten for this purpose is that man happens to have ten fingers on his two hands. The purpose of this chapter is to make clear the principle of place value and how it contributes to our ease in handling numbers.

We are so familiar with our own system of numeration using base ten that we sometimes fail to sense clearly that it is but one of a broad class of numeration systems all of which have the same feature of place value but use different bases. It may help to see the features of our place value system if we study others with different bases. We have selected the system with seven as its base for this purpose.

The characteristic of any place value numeration system is the idea of grouping and the use of a symbol in a certain position in a numeral to represent the number of groups of a certain size corresponding to that position. Thus when the base is ten the groups represent units or tens or hundreds, etc., and the numeral "243" means two hundreds and four tens and three units. Since the grouping is by tens in this system, its base is ten and we call the system a decimal system from the Latin word decem which means ten.

Note that while we use special words like "hundred," "thousand," "million," etc., to represent the size of certain groups the important thing to realize is that for any given symbol each place to the left has ten times the value of the given place. The first place tells us how many groups of one. The second place tells us how many groups of ten, or ten times one (10 × 1). The third place tells us how many groups of ten times ten (10 × 10), or one hundred; the next, ten times ten times ten (10 × 10 × 10), or one thousand, and so on. By using a base and the ideas of place value, it is possible to write any number in the decimal system using only ten basic symbols, for example, the digits 0, 1, 2, ..., 9. There is no limit to the size of numbers which can be represented by the decimal system.

If now instead of grouping by ones, tens, and ten tens, we use groups of ones, fives, and five fives or ones, sevens, and seven sevens, we would be using place value systems with bases five or seven respectively.

We saw that the numeral 243 in the decimal system means 2 hundreds + 4 tens + 3 units or 2 groups of ten tens + 4 groups of ten + 3 units. If the base were seven, what would the numeral 243 mean? It would mean 2 groups of seven sevens + 4 groups of sevens + 3 units. We write it as 243_{seven} where the subscript "seven" tells us that we are using seven as the base. We would not write the numeral as 243_7 because the numeral 7 is not used in this base.

What would 243_{seven} be if it were converted to base ten?

$$\begin{aligned} 243_{\text{seven}} &= (\underline{2} \times [7 \times 7]) + (\underline{4} \times [7]) + (\underline{3} \times [1])^* \\ &= (2 \times 49) + (4 \times 7) + 3 \\ &= 98 + 28 + 3 \\ &= 129_{\text{ten}} \end{aligned}$$

If we use five as a base for grouping, the five basic symbols are the numerals 0, 1, 2, 3, 4.

What would 243_{five} be if it were converted to base ten?

$$\begin{aligned} 243_{\text{five}} &= (2 \times [5 \times 5]) + (4 \times [5]) + (3 \times [1]) \\ &= 50 + 20 + 3 \\ &= 73_{\text{ten}} \end{aligned}$$

* When we wish to indicate grouping in a mathematical sentence, we use parentheses:

$$(7 + 10) + 3 = 17 + 3 = 20$$

When we wish to indicate further groupings within the parentheses, we use brackets to keep the groupings clearly indicated. Thus:

$$129 = (1 \times [10 \times 10]) + (2 \times [10]) + (9 \times [1])$$

Once again

$$
\begin{aligned}
2030_{seven} &= (2 \times [7 \times 7 \times 7]) + (0 \times [7 \times 7]) + (3 \times [7]) + (0 \times [1]) \\
&= (2 \times 343) + 21 \\
&= 686 + 21 \\
&= 707_{ten}
\end{aligned}
$$

While

$$
\begin{aligned}
2030_{five} &= (2 \times [5 \times 5 \times 5]) + (0 \times [5 \times 5]) + (3 \times [5]) + (0 \times [1]) \\
&= (2 \times 125) + 15 \\
&= 265_{ten}
\end{aligned}
$$

Notice the importance of the symbol for zero in any place value system. Without it we would not know whether 23 meant 23 or 203 or 230. In fact, it is the absence of such a symbol in the ancient Babylonian numeration system that makes the translation of their mathematical writings so difficult.

Let us look a little more closely at the systems whose bases are five and seven. How do we write numerals in these systems? What numerals in base five and seven represent the number of x's in each of the following two sets?

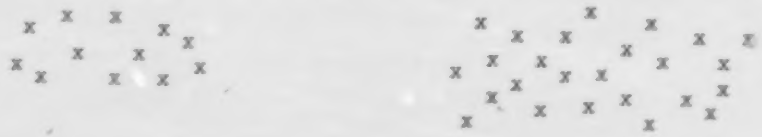


Figure 4-1.

We group these by fives in Figure 4-2.

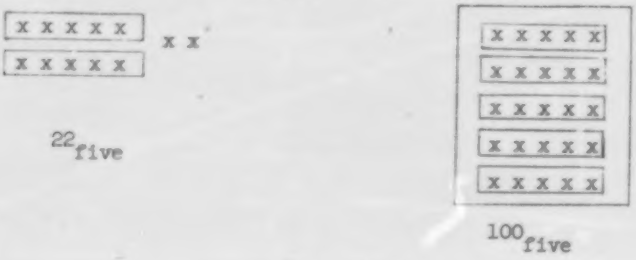
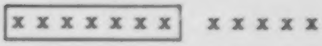


Figure 4-2.

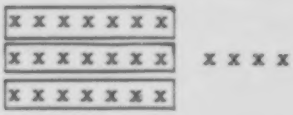
This shows that the numerals may be written 22_{five} and 100_{five}.

We group the same two sets by sevens in Figures 4-3 and 4-4.



15_{seven}

Figure 4-3.



34_{seven}

Figure 4-4.

Thus the numbers may be written in base seven as 15_{seven} and 34_{seven}. Written in base ten the numbers are of course 12 and 25.

In more detail we see that in Figure 4-3 there is one group of seven and five more. The numeral is written 15_{seven}. In this numeral, the 1 indicates that there is one group of seven, and the 5 means that there are five ones.

In Figure 4-4 how many groups of seven are there? How many x's are left ungrouped by sevens? The numeral representing this number of x's is 34_{seven}. The 3 stands for three groups of seven, and the 4 represents four single x's or four ones.

When grouping is by sevens the number of individual objects left can only be zero, one, two, three, four, five, or six. Symbols are needed to represent those numbers. Suppose the familiar 0, 1, 2, 3, 4, 5, and 6 are used for these rather than new symbols. No other symbol is needed for the base seven system.

If the x's are marks for days, 15_{seven} is a way of writing one week and five days. In our decimal system this number of days is "twelve" and is written "12" to show one group of ten and two more.* We should not use the name "fifteen" for 15_{seven} because fifteen is 1 ten and 5 more. We shall simply read 15_{seven} as "one, five, base seven."

---

* The base name in our decimal numerals is not written since the base is always ten. From now on any numeral without a subscript will be considered a base ten numeral.

Notice that in base ten, zero, one, two, three, four, five, six, seven, eight, and nine are represented by single symbols. The base number "ten" is represented as 10, and it means one group of ten and zero more.

With this in mind, think about counting in base seven. The next numeral after 6_{seven} would be 10_{seven}, that is, one seven and no ones. What would the next numeral after 66_{seven} be? Here you would have 6 sevens and 6 ones plus another one. This equals 6 sevens and another seven, that is, seven sevens. How could seven sevens be represented without using a new symbol? A new group is introduced, the seven sevens group. This number would then be written 100_{seven}. What number does this numeral name? The answer is given below. Go on from this point and write a few more numerals. What would be the next numeral after 666_{seven}?

Now you are ready to write a list of place values for base seven.

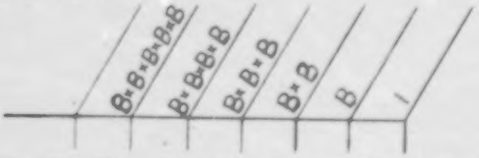
Place Values for Base Seven

$7 \times (7 \times 7 \times 7)$	$7 \times (7 \times 7)$	7×7	7×1	1
7×343	7×49	7×7	7×1	1
2401	343	49	7	1

Figure 4-5.

Notice that each place represents seven times the value of the next place to the right. The first place on the right is the units place in the system with base seven as well as in that with base ten. This is characteristic of all place value numeration systems. The value of the second place is the base times one which is seven. The value of the third place from the right represents a group of (seven × seven) or forty-nine, and in the next place (seven × seven × seven).

In general, for any base B the value of positions may be illustrated:



Now what is the $\text{numeral}_{\text{ten}}$ for the number represented by 100_{seven}? It is (1 × [seven × seven]) + (0 × seven]) + (0 × one) = forty-nine, since forty-nine is written in base ten as 49, then $100_{\text{seven}} = 49$.

In like manner we see that

$$246_{seven} = (2 \times \text{seven} \times \text{seven}) + (4 \times \text{seven}) + (6 \times \text{one}).$$

Figure 4-6 shows the actual grouping represented by the digits and the place values in the numeral 246_{seven}.

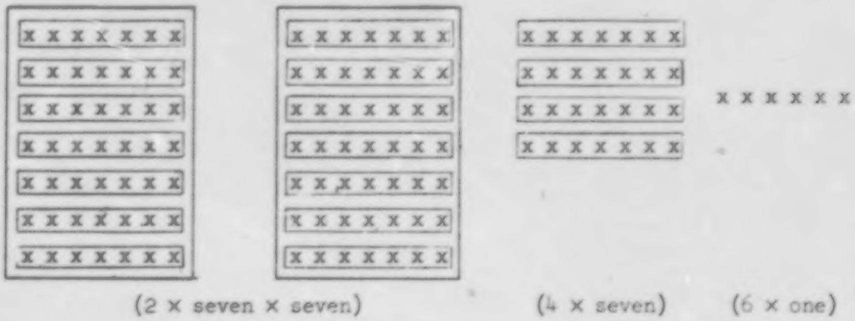


Figure 4-6.

If we wish to express the number of x's above in the decimal system of notation, we may write:

$$\begin{aligned} 246_{seven} &= (2 \times 7 \times 7) + (4 \times 7) + (6 \times 1) \\ &= (2 \times 49) + (4 \times 7) + (6 \times 1) \\ &= 98 + 28 + 6 \\ &= 132 \end{aligned}$$

Problems[*]

1. How would you write the numeral in the base seven for the number 45? Illustrate by showing groupings of "x's" or dots.
2. Now consider 111 and make a diagram showing how many groups of sevens and seven sevens there are. What numeral is this in base seven?

Summary

The essential components of any positional numeration system are: a base, for example, base seven; a zero; ordered symbols, for example, 0, 1, 2, 3, 4, 5, 6; and place value using successive multiples of the base, for example, 1, 7, (7 × 7), (7 × 7 × 7), (7 × 7 × 7 × 7), etc. Note that the numeral for the base within the system is always 10, that is, 1 of the base and zero ones. Thus in our decimal system, "ten" is 10; in base seven system "seven" is 10_{seven}; in base twelve system "twelve" is 10_{twelve}, etc. In general, if a numeral is written in a base B system, the first symbol on the right indicates the number of 1's, the next place to the left indicates the number of B's, the next place to the left the number of B × B's, etc. Within this system B is written 10_B, B × B is written 100_B, B × B × B is written 1000_B, etc.

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[*] Solutions for problems are on page 38.

Exercises - Chapter 4

1. What is the base ten value of the "5" in each of the following numerals?

   a. 560_{seven} b. 605_{seven} c. 5060_{seven}

2. In the base seven system write the value of the fifth place counting left from the unit's place.

3. What numeral in the base seven system represents the number named by six dozen?

4. On planet X-101, the pages in books are numbered in order as follows: |, └, △, ⊡, ⧅, ⊠, |−, ||, |└, |△, |□, |⧅, |⊠, └−, └|, etc. What seems to be the base of the numeration system these people use? Why? How would the next number after └| be written? Write numerals for numbers from □− to ⧅△.

5. Create a place value system where the following symbols are used.

Symbol	Decimal Value	Name
○	0	do
|	1	re
∧	2	me
Z	3	fa
|○	4	re-do

   Write the numerals for numbers from zero to twenty in this system.

6. Group a set of objects so that you will have sets of eight fives and two ones. What is the numeral which represents this grouping?

7. Group the same set of objects so that you will have sets of tens. What is the numeral which represents this grouping?

8. Now group the same set of objects to show sets of seven. What is the numeral which represents this grouping?

9. When grouping in fives, we use place value and symbols to name the numerals. What are the symbols needed in order to write numerals in the base five system? In the base seven system? In the base eight system?

10. Write the base five numeral for each of these items.

a. $9 = \underline{\hspace{2cm}}_{\text{five}}$

b. $24 = \underline{\hspace{2cm}}_{\text{five}}$

c. $32 = \underline{\hspace{2cm}}_{\text{five}}$

d. $15 = \underline{\hspace{2cm}}_{\text{five}}$

e. $3 = \underline{\hspace{2cm}}_{\text{five}}$

f. $22 = \underline{\hspace{2cm}}_{\text{five}}$

11. Now write the base seven numeral for the above items.

12. Write in base ten the numeral representing the number of objects which is meant by each of these base five and base seven numerals.

a. $23_{\text{five}} = \underline{\hspace{1.5cm}}$

b. $13_{\text{five}} = \underline{\hspace{1.5cm}}$

c. $2_{\text{five}} = \underline{\hspace{1.5cm}}$

d. $24_{\text{seven}} = \underline{\hspace{2cm}}$

e. $33_{\text{seven}} = \underline{\hspace{2cm}}$

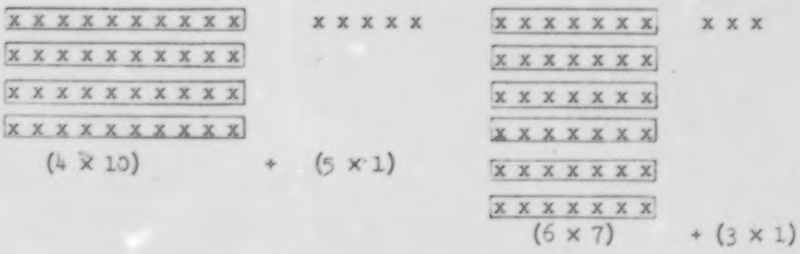
f. $41_{\text{seven}} = \underline{\hspace{2cm}}$

13. The following numbers are written using base ten. If you were to write each number using first the base three system, then the base seven system, which would require more digits?

a. 7
b. 10
c. 12
d. 2
e. 5
f. 3

14. Repeat Problem 13 using base two and base five.

Solutions for Problems

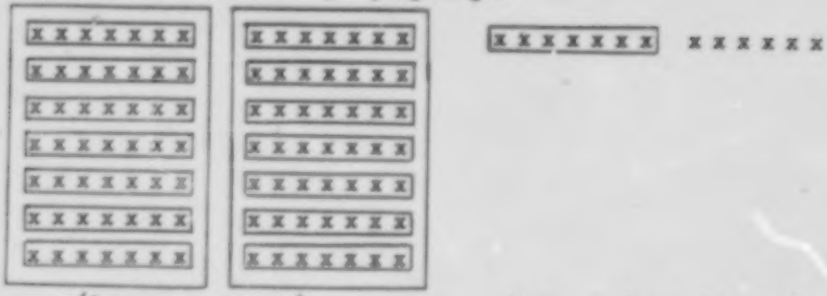


$45 = 63_{\text{seven}}$

2. 111 in the decimal system means

$$(1 \times 100) + (1 \times 10) + (1 \times 1)$$

or using seven as a basis for grouping we get



$$(2 \times \text{seven} \times \text{seven}) \quad + \quad (1 \times \text{seven}) \quad + \quad (6 \times \text{one})$$

Thus 111 = 216_{seven}.

Chapter 5

PLACE VALUE AND ADDITION

In the last chapter we noted that we ordinarily combine the place value idea with the use of ten as a base to get our decimal system of numeration. By using a base and the ideas of place value, it is possible to write any number in the decimal system using the ten symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. We also indicated that ten is probably used as a base because man has ten fingers and it is natural for primitive people to count by making comparisons with their fingers. If man had six or eight fingers, he might have counted by sixes or eights. Since he does have twenty fingers and toes he sometimes counts by twenties.

By investigating other number bases, we became more aware of how our own system works. If, instead of grouping by tens and ten tens, we use groups of fives and five fives or sevens and seven sevens, we are using place value systems with bases five and seven respectively. In general, we can use any whole number greater than one as the base of a number system. If we use $\underline{b}$ as the base, then the first place on the right still indicates the number of ones in the number, but the second place is used to tell how many groups of $\underline{b}$, the third place how many groups of $b \times b$, etc. If five is used as the base, we need only the digits 0, 1, 2, 3 and 4 to write any number; if three is used, we need only 0, 1 and 2.

In a numeration system with a place value principle, the base of the system determines the number of digits to be used in writing numerals in that system. A numeration system with base twelve has been advocated and in fact when we buy eggs by the dozen and pencils by the gross (twelve dozen) we are using just such a system. The main disadvantage of this system is that it would need two new digits to represent ten and eleven. We invent the symbols X (read dec) and ∈ (read el) for use when we want to discuss this system later on.

We will write a few numbers in the base three system to review the pattern. We start with 0, 1, 2. The next number is written 10 to indicate that we have 1 group of three and no units. Then 11, 12, 20, 21, 22. This last numeral means 2 groups of three and 2 units; this is eight. The next number would be nine which is 1 group of three threes and no threes or units and so would be written as 100_{three}.

For example, 1221 as a base three numeral would be:

$$\begin{aligned}1221_{\text{three}} &= (\underline{1} \times [3 \times 3 \times 3]) + (\underline{2} \times [3 \times 3]) + (\underline{2} \times 3) + (\underline{1} \times 1)\\ &= 27 + 18 + 6 + 1 \quad \text{or} \quad 52.\end{aligned}$$

A base two or binary system is of interest to us because it is used by some modern, high-speed computing machines. These computers, sometimes incorrectly called "electronic brains," use the base two because an electric switch has only two positions, "on" or "off" and these two positions can be used to represent the two digits 0 and 1 of the binary system.

The numeral 11010 in base two stands for

$$(\underline{1} \times [2 \times 2 \times 2 \times 2]) + (\underline{1} \times [2 \times 2 \times 2]) + (\underline{0} \times [2 \times 2]) + (\underline{1} \times 2) + (\underline{0} \times 1)$$
$$\text{or} \quad 16 + 8 + 2 \quad \text{or} \quad 26.$$

Counting in the base two system would go: 0, 1, 10, 11, 100, 101, etc.

A chart such as the following one is helpful in seeing better the numeral sequence for place value numeration systems with different bases.

BASE

Twelve	Ten	Eight	Seven	Five	Three	Two
1	1	1	1	1	1	1
2	2	2	2	2	2	10
3	3	3	3	3	10	11
4	4	4	4	4	11	100
5	5	5	5	10	12	101
6	6	6	6	11	20	110
7	7	7	10	12	21	111
8	8	10	11	13	22	1000
9	9	11	12	14	100	1001
X	10	12	13	20	101	1010
E	11	13	14	21	102	1011
10	12	14	15	22	110	1100
11	13	15	16	23	111	1101
12	14	16	20	24	112	1110
13	15	17	21	30	120	1111
14	16	20	22	31	121	10000
15	17	21	23	32	122	10001
16	18	22	24	33	200	10010
17	19	23	25	34	201	10011
18	20	24	26	40	202	10100
19	21	25	30	41	210	10101
1X	22	26	31	42	211	10110
1E	23	27	32	43	212	10111
20	24	30	33	44	220	11000
21	25	31	34	100	221	11001

As seen from the chart, the base numeral always appears as 10 when written in that particular base system. Similarly, in a particular base system the numeral 100 always designates the number obtained by multiplying the base by itself. We should read the numeral 10_{seven} as "one zero, base seven" not as ten, in order to avoid unnecessary confusion.

Figure 5-1.

Numbers may be expressed by different numerals. For example, twelve may be written as ∩||, XII, 12_{ten}, 15_{seven}, or 1100_{two} and so on. These *numerals* are not the same, yet they represent the *same number*. The symbols used are not in themselves numbers. "XII" is not twelve things nor is "10_{twelve}." They are only different numerals, or symbols for twelve. In fact, even "twelve" is a symbol for twelve.

Properties of Numbers

We have been studying what numbers are, how they may be symbolized by different kinds of numerals, and how they may be represented on a number line. Let us now consider some of the properties of the operations used in combining numbers and the results from using different operations of combining given numbers. Are there rules which numbers obey or is each number a law unto itself? Are there numbers which merit special attention because they act in unique ways?

The four ways of combining numbers with which you are most familiar are addition, subtraction, multiplication and division. In this and the next two chapters addition and subtraction will be studied. In Chapters 8, 9, 10 and 11 multiplication and division will be analyzed and explained.

Since numbers were defined in terms of sets of objects, we will go back to the idea of sets in order to get started in this new investigation. Let us consider a pair of sets which have no elements in common. If set A has no elements in common with set B, we express this idea by saying that A and B are *disjoint sets*.

There is a standard way of making a new set out of a pair of such sets. For example: if we have the set of boys in a classroom and the set of girls in a classroom, we can join these two sets and get the set of pupils in the classroom. If A is the set of boys and B is the set of girls, the set of boys and girls is written $A \cup B$ and is read as "A *join* B," or "A *union* B" or the *join* of A and B. We use this operation of joining only if A and B are disjoint; in other words, only if no element belongs to both A and B. Another example of the joining of two sets: if A is the set of frogs in a pond and B the set of turtles in a pond, then $A \cup B$ is the set containing frogs and turtles in the pond.

The notion of joining sets is the basis for the notion of adding numbers and the basic facts about the joining of sets underlie certain arithmetical facts. To take one property that has a direct link to addition, if A and B are sets, then $A \cup B$ is the same as $B \cup A$. That is, it doesn't matter which comes first and which comes second in performing the join operation. We say that the join operation is commutative. In the example of the frogs and turtles, $A \cup B$ and $B \cup A$ is the same set of turtles and frogs in the pond.

A second property of the join of two sets that has a direct link to addition indicates how we can proceed if we have three sets, for we know that the join operation can only work on two sets at a time. If A, B and C are sets, then because $A \cup B$ is a set, we may consider $(A \cup B) \cup C$ as the join of $(A \cup B)$ and C and because $B \cup C$ is a set we may consider $A \cup (B \cup C)$ as the join of A and $(B \cup C)$. The interesting fact is that either way we consider the various joins, the end result is the same, that is, $(A \cup B) \cup C = A \cup (B \cup C)$ and we say that joining is associative. Thus, if A is a set of nails, B is a set of screws and C is a set of tacks, then $(A \cup B) \cup C$ is the set of nails and screws joined with the set of tacks and $A \cup (B \cup C)$ is the set of nails joined to the set of screws and tacks. In both cases the resulting set is the same.

If A is the set of brown cows and B is the set of purple cows, then $A \cup B$ is still the set of brown cows since B (the set of purple cows) is a set that has no members. A set which has no members is called the empty set. The set of all boys eight feet tall in a third grade classroom; the set of all three-year old children in college or the set of all crocodiles in the Yukon River are all examples of the empty set. Children delight in using their imaginations to make up examples of the empty set.

How does the notion of the "join of two sets" help us understand the idea of addition? Addition is essentially an operation on two numbers: given two numbers $\underline{a}$ and $\underline{b}$ we can always associate with them a third number $\underline{c}$. The problem is how to determine $\underline{c}$. In Chapter 3 we learned how to count members of a set and thus how to determine the number property of the set. If we have two numbers $\underline{a}$ and $\underline{b}$, we can choose a set A having $\underline{a}$ members and a disjoint set B having $\underline{b}$ members. Now consider the join of A and B which is the set $A \cup B$. By definition:

If $N(A) = a$ and $N(B) = b$, then
$a + b$ is the number of members in $A \cup B$.

In order to illustrate this definition in a concrete manner, let us consider the case of adding 3 and 4. We look for two sets A and B which have the number properties we want. A could be the set {✡, ○, △} and B, the set {□, ✱, ◇, ∂}. We first check that A and B are disjoint sets. Then A ∪ B = {✡, ○, △, □, ✱, ◇, ∂}. By counting its members we see that the number property of this set is 7. Remember the number property of any set A is simply the number of elements in the set and is written $N(A)$. In our example we found disjoint sets A and B such that $N(A) = 3$, $N(B) = 4$, and then we found that $N(A \cup B) = 7$; hence, by our definition $3 + 4 = 7$.

It is important to remember that addition is an operation on two numbers, while joining is an operation on two sets. We put together or join two sets to form a third set, while we add two numbers to get a third number. Since addition of two numbers is defined in terms of the join of two sets, we may determine the properties of addition by considering the properties of joining sets. By giving children many experiences in joining sets and determining the number properties involved, we can teach not only the properties of addition, but the addition facts such as $2 + 2 = 4$, $2 + 3 = 5$, $3 + 4 = 7$, etc.

The first important property of addition is simply that we can always do it. That is, if we add two whole numbers, we always get a whole number. Technically, we say that the set of whole numbers is *closed* under the operation of addition. This property is called the *closure* property. Observe that subtraction does not have this property.

What other properties of addition can we get from our definition that $a + b$ is the number of members in $A \cup B$? To find $\underline{b} + \underline{a}$ we have to consider the number of members of $B \cup A$. We already know that $B \cup A$ is precisely the same set as $A \cup B$. Therefore it is true that for any whole numbers $\underline{a}$ and $\underline{b}$,

$$a + b = b + a .$$

This property of addition may seem so obvious that it is scarcely worth mentioning, but like the property of closure it is a property that subtraction does not have. For example, $8 + 2 = 2 + 8$ but does $8 - 2 = 2 - 8$? We name this property the *commutative property of addition* or we may say, "addition is commutative."

If addition is an operation on two numbers, how can we add three numbers *a*, *b* and *c* ? We add two of them together and then to the result we add the third number. Is it important which two we add first? Going back to the definition of addition in terms of the union of sets, we find that $(a + b) + c$ is the number of members in the set $(A \cup B) \cup C$ while $a + (b + c)$ is the number of members in the set $A \cup (B \cup C)$. Since these two sets are identical, the number we get either way is the same: $(4 + 6) + 7 = 10 + 7 = 17$ but also $4 + (6 + 7) = 4 + 13 = 17$. This property is called the *associative property of addition* or we may say, "addition is associative."

That is, for any three numbers a, b and c

$$(a + b) + c = a + (b + c).$$

Repeated use of the associative and commutative properties enables us to group numbers for addition in the most convenient form. To add $7 + 6 + 3 + 2 + 4$ we may first use the commutative property to say that $6 + 3 = 3 + 6$ and $2 + 4 = 4 + 2$. Now we associate the first two numbers and the second two numbers to get $(7 + 3) + (6 + 4) + 2 = (10 + 10) + 2 = 22$.

There is one whole number which plays a special role with respect to addition and that is the number zero. If we join the empty set to any set A we still have set A. Therefore we see that $0 + a = a + 0 = a$. Since the addition of zero to any number leaves that number identical, we say that zero is the *identity element* with respect to addition. Zero is the only number with this interesting property.

Addition on the Number Line

Another useful way of thinking about addition is to consider it with respect to the representation of numbers on the number line. Draw a number line with the point 1 lying to the right of the point 0.

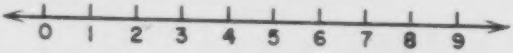


Figure 5-2. A number line.

To add the number 3 to the number 4 start from 0, move four units to the right to the number 4, then move three more units to the right from the number 4. We stop at 7 so $4 + 3 = 7$. Although we are using the specific numbers 3 and 4 for our example, this process will work for any numbers *a* and *b* and illustrates the closure property.

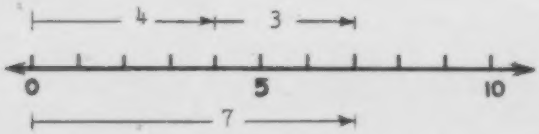


Figure 5-3. Closure property: $4 + 3 = 7$

The commutative property may be illustrated on the number line.

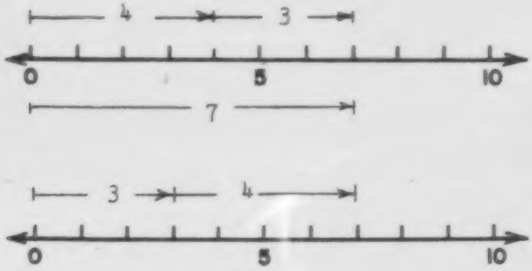


Figure 5-4. Commutative property: $4 + 3 = 3 + 4$

The associative property can also be illustrated on the number line, though the process is a bit more involved. Figure 5-5a shows $(3 + 5) + 4$ by first showing $(3 + 5)$, then taking the result and adding 4 to it. Figure 5-5b, on the other hand, shows $3 + (5 + 4)$ by first showing $(5 + 4)$, then taking that result and adding it to 3. The dotted segments show the result of $(5 + 4)$ being moved down to be added to 3.

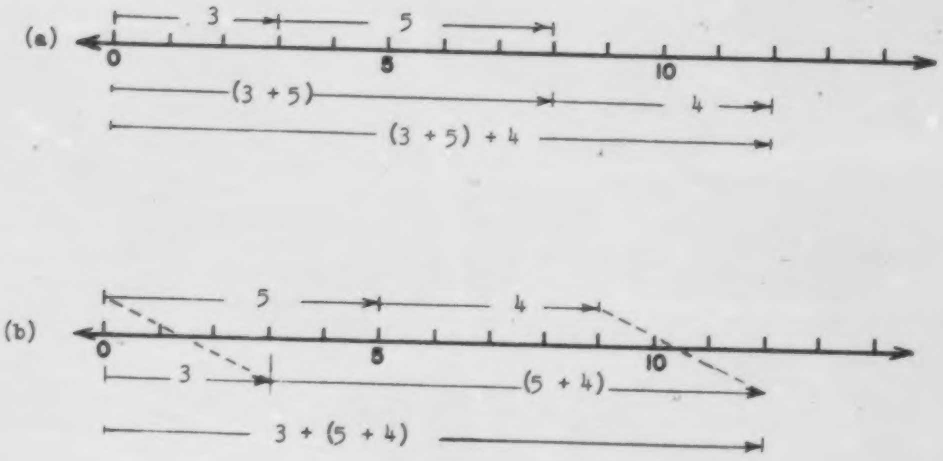


Figure 5-5 . Associative property: $(3 + 5) + 4 = 3 + (5 + 4)$

If the number line is used frequently to illustrate addition of whole numbers, the familiarity with its properties will help a great deal in working with numbers and in answering questions about numbers.

Exercises - Chapter 5

1. A = {dog, cat, cow, pig} and
   B = {duck, horse, elephant}
   What is $A \cup B$?
2. R = {2, 4, 6, 8, 10, 12} and
   S = {1, 3, 5, 7}
   What is $R \cup S$?
3. If P is the set of all white horses, and G is the set of all lavender horses, what is set $P \cup G$?
4. Draw number lines to show the following addition examples.
   - a. $3 + 5 = 8$
   - b. $9 + 2 = 11$
   - c. $4 + 8 = 12$
   - d. $2 + 6 = 8$
   - e. $(3 + 5) + 7 = 15$
   - f. $3 + (5 + 7) = 15$

5. Which of the following mathematical sentences are direct examples of the commutative property of addition?
   a. $18 + 11 = 11 + 18$
   b. $203 + 401 = 200 + 404$
   c. $6 + 7 = 5 + 8$
   d. $8 + (7 + 3) = (8 + 7) + 3$
   e. $1,207 + 109 = 109 + 1,207$

6. By which property of addition are each of the following mathematical sentences true?
   a. $(2 + 3) + 4 = 2 + (3 + 4)$
   b. $(18 + 19) + (39 + 12) = (39 + 12) + (18 + 19)$
   c. $(8 + 9) + 6 = 6 + (8 + 9)$
   d. $(8 + 9) + 6 = 8 + (9 + 6)$
   e. $(8 + 9) + 6 = (9 + 8) + 6$

7. Consider the mathematical sentence $(6 - n) - 1 = 6 - (n + 1)$
   a. What is the largest whole number n which makes this sentence true?
   b. What is the smallest whole number n which makes this sentence true?
   c. Find all the whole numbers n which make this sentence true.

8. For each of the mathematical sentences below decide which whole numbers can make the sentence true. There may be no answer, one answer, or more than one answer.
   a. $n - 1 = 1 - n$
   b. $n - 10 = 10 - n$
   c. $6 + n = n + 6$
   d. $n + 50 = 50 + n$
   e. $10 + n = 10$
   f. $9 - n = 9$
   g. $n = n - 1$

9. Use number lines to illustrate the following examples of the associative property of addition.
   a. $(2 + 3) + 4 = 2 + (3 + 4)$
   b. $(5 + 1) + 2 = 5 + (1 + 2)$

10. Let $A = \{\text{bicycle, telephone, radio, airplane}\}$ and
    $B = \{\text{canoe, diving board, lake}\}$
    a. What is the number property of A ?
    b. What is the number property of B ?
    c. What is the number property of $A \cup B$?

11. What advantages or disadvantages, if any, do the binary and duodecimal system have as compared to the decimal system?

12. People who work with high speed computers sometimes find it easier to express numbers in the octal, or eight system rather than the binary system. Conversions from one system to the other can be done very quickly. Can you discover the method used?

    Make a table of numerals as shown below:

Base ten	Base eight	Base two
1	1	1
2	2	10
5	5	101
7	?	?
15	?	?
16	?	?
32	?	?
64	?	?
256	?	?

Compare the powers of eight and two up to 256. Study the powers and the table above. $101,011,010_{two} = 532_{eight}$. Can you see why?

13. An inspector of weights and measures carries a set of weights which he uses to check the accuracy of scales. Various weights are placed on a scale to check accuracy in weighing any amount from 1 to 16 ounces. Several checks have to be made, because a scale which accurately measures 5 ounces may, for various reasons, be inaccurate for weighings of 11 ounces and more.

    What is the smallest number of weights the inspector may have in his set, and what must their weights be, to check the accuracy of scales from 1 ounce to 15 ounces? From 1 ounce to 31 ounces?

Chapter 6

SUBTRACTION AND ADDITION

Introduction

In the preceding chapter we dealt with the fundamental connection between the union of sets and the addition of whole numbers. This fundamental connection is:

$$N(A) + N(B) = N(A \cup B),$$

whenever A and B are disjoint sets. Here N(A) is the number of elements in A or the number property of A, and N(B) is the number of elements in B.

We now come to the operation of subtraction of whole numbers. This operation is more complicated, conceptually, than addition. There are two fundamentally different approaches to subtraction. The first approach starts with set operations and defines the operation of subtraction of numbers in terms of these operations. The second approach is more abstract, and defines subtraction directly in terms of addition of whole numbers.

To make things even more complicated, the first approach, the more concrete one, can be done in two different ways, thus yielding three different ways of thinking about subtraction.

In the classroom, no sharp distinction is made between these different approaches, but it is important that the teacher realize what the differences are in order to guide learning more effectively. We shall therefore discuss these approaches one by one. In order to do this, we first need to discuss two more concepts related to sets, namely "subset" and "remainder set."

Subsets

So far we have only considered disjoint sets. Now we consider a different case. Suppose we have a set A and another set B all of whose members are also members of A. Then we say that B is a subset of A. An illustration of this is:

A = {all the members of a class of children}
B = {all the boys in the class of children}

Each member of B is a member of A, so B is a subset of A.

Here is another example:

A = {○,△,□,★}

Some of the subsets of A are

{○,△}
{△,□,★}
{□}

Is the set B = {○,□,ℰ} a subset of A ? The answer is no, because there is at least one member of B, namely ℰ, which is not a member of A.

More generally we see that

> B is a subset of A if every member of B is a member of A.

Another way to say this is

> B is a subset of A if there is no member of B which is not also a member of A.

Both statements say exactly the same thing.

The second statement above leads us to accept the statement that the empty set is a subset of the set A, since there is no member of the empty set which is not a member of A. In fact, the empty set is a subset of every set.

Notice also that according to our definition of subset, any set is a subset of itself, for it is surely true that every member of A is a member of A.

Thinking in terms of a set of numbers, let

A = {2, 4, 6}

We can write all the subsets of A :

A = {2, 4, 6}
B = {2, 4}
C = {2, 6}
D = {4, 6}
E = {2}
F = {4}
G = {6}
H = { }

Remainder Sets

Suppose we have a set A and another set B which is a subset of A. Now consider the set which consists of all the elements of A which are *not* elements of B. We call this new set a *remainder set* and denote it by $A \sim B$. For example, if A is the set of all the children in your class and B is the set of boys in your class, then $A \sim B$ is the set of girls in your class. Similarly, if $A = \{\bigcirc, \triangle, \square, \star\}$ and $B = \{\bigcirc, \square\}$, then $A \sim B = \{\triangle, \star\}$. Also, $A \sim A = \{\ \}$, the empty set. Note that removing the elements of subset B from A is indicated by the symbol "~", read "wiggle," and not by the familiar sign "−" which we reserve for the subtraction of two numbers.

We now have two set operations, that of forming a union and that of forming a remainder set. The fundamental connection between them is:

$$(A \sim B) \cup B = A$$

In words: If we first form the remainder set $A \sim B$ and then form the union of it with B, we get back the original set A. For example, if A is the set of children in your class, and if B is the set of boys, then $A \sim B$ is the set of girls in the class, and the union of $(A \sim B)$ and B is the set of all girls and all boys, in other words, the whole class. We say that these two operations are *inverse*.

The idea of the inverse of an operation is a very important idea in mathematics, as it is in many non-mathematical situations, and it is an idea which you will meet again and again. If we wish to put the idea in a non-mathematical situation we might ask the question, What will "undo" taking two steps backward? The answer is taking two steps forward. The inverse of falling asleep is waking up; the inverse of putting on a coat is taking it off; the inverse of standing up is sitting down.

First Definition of Subtraction

We can use the idea of remainder set to define the operation of subtraction of whole numbers. If *a* is a number and if *b* is a number less than or equal to *a*, we first choose a set A such that $N(A) = a$. Next we pick a set B which is a subset of A and such that $N(B) = b$.

These two sets determine the remainder set $A \sim B$. The number of elements in $A \sim B$ is, of course, $N(A \sim B)$. Then the first definition of subtraction is

$$a - b = N(A \sim B).$$

For example, if $a = 5$ and $b = 2$, we can choose A to be the set

$$A = \{\bigcirc, \wedge, \square, \bigstar, \mathcal{E}\}.$$

Next we can choose B to be the subset

$$B = \{\wedge, \bigstar\}.$$

Then

$$A \sim B = \{\bigcirc, \square, \mathcal{E}\}.$$

Now our definition tells us that

$$5 - 2 = N(A \sim B) = 3.$$

Note that if we made a different choice for B, for example

$$B = \{\square, \mathcal{E}\},$$

the result would be the same. Also, if we had chosen a different set A, for example $A = \{V, W, X, Y, Z\}$, and any two member subset of this set as B, the result would still be the same.

Problem*

1. Use this definition of subtraction to compute in detail $7 - 3$.

Second Definition of Subtraction

This definition does not use the idea of remainder set, but uses the ideas of union of disjoint sets and of one-to-one matching. If $\underline{a}$ is a number and if $\underline{b}$ is a number, with $b \leq a$, we start by choosing a set A with $N(A) = a$ and a set $\overline{B}$ disjoint from A with $N(\overline{B}) = b$. ($\overline{B}$ is used for this set instead of B to remind us that it is a set disjoint from A and not a subset of A as B was in the first definition. In both cases the use of the letter B in the name reflects the fact that the set is chosen to have $\underline{b}$ members.)

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* Solutions for problems in this chapter are on page 65.

Next we choose a set C, disjoint from both A and $\overline{B}$ in such a way that A and $(\overline{B} \cup C)$ are in one-to-one correspondence. That is, there is a matching of the elements of A to the elements of $\overline{B} \cup C$. Then the second definition of subtraction is:

$$a - b = N(C).$$

In other words, having chosen appropriate disjoint sets A and $\overline{B}$ we look for a third set C with just the right number of members so that the union of this set and the set $\overline{B}$ will exactly match up with the set A. The number of members in such a set C tells us "how much larger" A is than $\overline{B}$.

As an example of this definition of subtraction let us again use $a = 5$ and $b = 2$. A can be the same set {○, ^, □, ☆, ê} as was used before, but $\overline{B}$ must now be a disjoint set with 2 members. Let $\overline{B} = \{X, Y\}$. Now if C is chosen to be the set {alpha, beta, gamma} it appears to be a perfect choice to fit the definition because $\overline{B} \cup C$ can be put into one-to-one correspondence with A. Such a matching is indicated in Figure 6-1 by arrows.

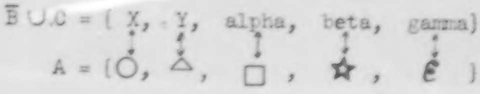


Figure 6-1. A matching of $\overline{B} \cup C$ and A.

Now by the second definition $5 - 2 = N(C) = 3$.

Problem

2. Use the second definition of subtraction to compute in detail $7 - 3$.

The most important thing to say about this definition of subtraction is that it always gives exactly the same result as the first definition We can illustrate this with the example, 5 - 2, used above.

First Definition — Second Definition

We choose the same set A in both cases.

$A = \{\bigcirc, \triangle, \square, \star, \mathcal{E}\}$ — $A = \{\bigcirc, \triangle, \square, \star, \mathcal{E}\}$

In the second definition we choose $\bar{B} = \{X, Y\}$

and $C = \{\text{alpha, teta, gamma}\}$

In the matching of Figure 6-1, the subset of A which matches the elements of $\bar{B}$ is $\{\bigcirc, \triangle\}$

$$\begin{array}{l}
\bar{B} \cup C = \{\overbrace{X,\ Y,}^{\bar{B}}\ \overbrace{\text{alpha, beta, gamma}}^{C}\} \\
\qquad\quad\ \ \updownarrow\ \ \updownarrow\quad\ \ \updownarrow\qquad\ \updownarrow\qquad\ \ \updownarrow \\
A = \{\bigcirc,\ \triangle,\ \ \square\ ,\ \ \star\ ,\ \ \mathcal{E}\ \}
\end{array}$$

To apply the first definition we now take this subset as B.

i.e., $B = \{\bigcirc, \triangle\}$.

Then, $A \sim B = \{\square, \star, \mathcal{E}\}$ which in the matching of Figure 6-1 is in one-to-one correspondence with the elements of C.

Therefore, $N(A \sim B) = N(C)$

But by Definition 1

$a - b = N(A \sim B)$ and by Definition 2 $a - b = N(C)$

Therefore, the second definition gives precisely the same result as the first definition.

Problem

3. Show by following the pattern of the preceding paragraph that there is a good reason for the fact that the results in Problems 1 and 2 are the same.

Now the question naturally arises as to why we should bother with two different definitions if they both give the same result. Why not use just one of them?

The reason is that there are two quite different kinds of problems that we commonly meet and it is important to know that the same mathematical operation can be used to solve both kinds of problems.

The first kind is the "take away" type:

> "John has 5 dollars and loses two of them.
> How many does he have left?"

The second kind is the "how many more" type:

> "John has 5 dollars. Bill has 2 dollars.
> How many more dollars does Bill need in order
> to have as many as John?"

The first definition of subtraction fits very well with the "take away" type of problem, and the second fits very well with the "how many more" type. But in each case the problem is solved by means of the subtraction: $5 - 2 = 3$.

Third Definition of Subtraction

In this definition we do not use sets at all, but instead work directly with whole numbers and the operation of addition.

Our definition is:

$a - b$ is the number $\underline{n}$ for which $b + n = a$.

Another way of putting this is:

The statements

$$a - b = n$$

and

$$b + n = a$$

mean exactly the same thing.

From this point of view, subtraction is the operation of finding the unknown addend, $\underline{n}$, in the addition problem

$$b + n = a.$$

For example we know that:

$$5 - 2 = 3 \quad \text{because} \quad 2 + 3 = 5.$$

Also, since we know that both

$$8 + 6 = 14 \quad \text{and} \quad 6 + 8 = 14,$$

we get both

$$14 - 6 = 8 \quad \text{and} \quad 14 - 8 = 6.$$

In general any addition fact gives us two subtraction facts automatically.

Problem

4. The two statements $a - b = n$ and $b + n = a$ mean the same thing. Working with whole numbers 6, 4 show the related addition and subtraction facts.

There are two reasons why it is important for teachers to understand this way, as well as the first two, of thinking about subtraction. The first is that this is the way that children usually think when they are developing their skills in computation. The second is that as children move through school, and study other kinds of numbers, such as fractions, decimals, negative numbers, etc., they will meet this idea of defining subtraction in terms of addition again and again.

It is important to realize that all three definitions of subtraction are equivalent and yield the same properties.

Properties of Subtraction

The operation of subtraction has some important properties that are easily seen.

For any number a, $a - 0 = a$.
For any number a, $a - a = 0$.
For any numbers a and b,
with $a \geq b$, $(a - b) + b = a$
and $(a + b) - b = a$.

The first is true because $A \sim B = A$ if B is the empty set. The second is true because $A \sim A$ is always the empty set. The third is true because, as we saw above, $(A \sim B) \cup B = A$.

From the third statement we see that subtracting a number and adding the same number are inverse operations. One undoes what the other does. For example

$$(7 - 5) + 5 = 7$$

and

$$(7 + 5) - 5 = 7.$$

It is important to note that the operation of subtraction does not have some of the properties that the operation of addition has. We can always add two whole numbers, but we can't always subtract. For example, we can add 4 and 7, but 4 - 7 is not defined. The reason is that if we have a set A such that N(A) = 4, we cannot find any subset B of A which has 7 members. Consequently, subtraction is not closed.

Moreover, subtraction, unlike addition, is not commutative. It is true that

$$4 + 7 = 7 + 4.$$

But 4 - 7 is not even defined, although 7 - 4 is.

Also we see that subtraction is not associative. For example,

$$(9 - 5) - 3 = 4 - 3 = 1$$

but

$$9 - (5 - 3) = 9 - 2 = 7.$$

Problems

5. By the definition of subtraction we see that if $b + n = a$, then $n = a - b$ and that $(a - b) + b = a$. Which properties are exemplified by the following:
   (a) $(202 - 200) + 200 = 202$
   (b) $(y - x) + x = y$
   (c) $[(30 - 15) - 5] + 5 = 15$
   (d) $5 + 0 = 5$

6. Does the sentence $(5 - 7) + 7$ make sense? (See third item under Properties of Subtraction.)

*7. Show by the use of the properties of addition and subtraction that the following sentence is true:

   If $b \geq a$, $\quad a + (b - a) = b$.

   Check that it is true by using several pairs of numbers.

Subtraction on the Number Line

If we consider subtraction with respect to the representation of numbers on the number line, we can illustrate many of its important processes and properties.

What is the answer to 9 - 4 ? We start on the number line at 9 and take away or move to the left 4 units thus arriving at 5 which is our answer.

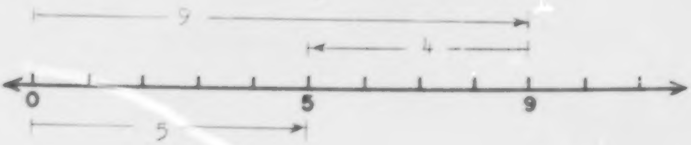


Figure 6-2. 9 - 4 = 5.

We know that 9 - 4 is the number n for which 4 + n = 9. This concept can be shown on the number line as follows where we see that n = 5 by bringing back the arrow representing n so that it starts at 0.

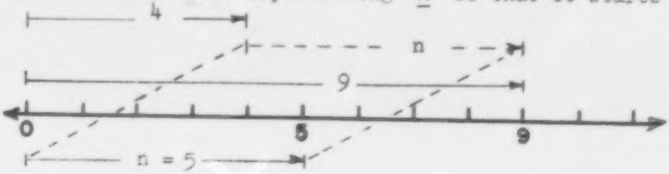


Figure 6-3. Since 4 + n = 9, 9 - 4 = n = 5.

We illustrated in Figure 5-5 the use of the number line to show the associative property of addition. Subtraction does not have the associative property for (13 - 5) - 2 = 8 - 2 = 6 while 13 - (5 - 2) = 13 - 3 = 10. In Figures 6-4 and 6-5 these problems are worked out on number lines. Figure 6-4a shows that 13 - 5 = 8 and this result is used in Figure 6-4b to get the answer 6. Similarly Figure 6-5a shows that 5 - 2 = 3 and this result is used in 6-5b to get the answer 10.

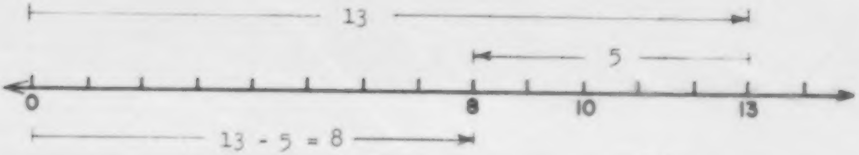

13
5
0
8
10
13
13 - 5 = 8

Figure 6-4a. 13 - 5 = 8.

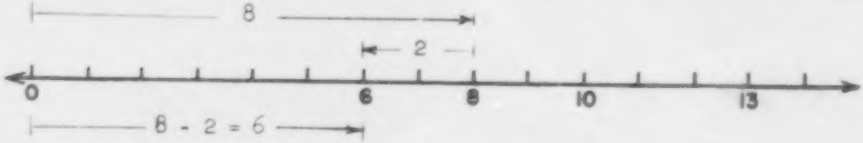

8
2
0
6
8
10
13
8 - 2 = 6

Figure 6-4b. 8 - 2 = 6.

Figure 6-4. Two number lines showing (13 - 5) - 2 = 8 - 2 = 6.

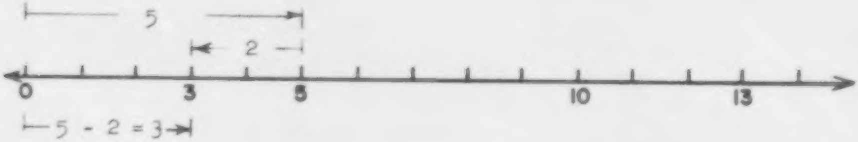

5
2
0
3
5
10
13
5 - 2 = 3

Figure 6-5a. 5 - 2 = 3.

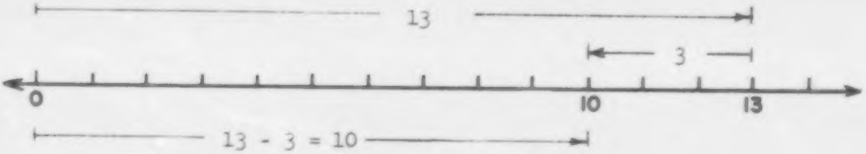
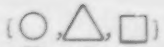
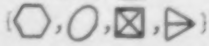
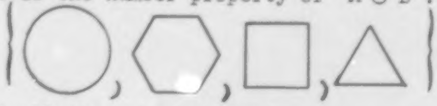

13
3
0
10
13
13 - 3 = 10

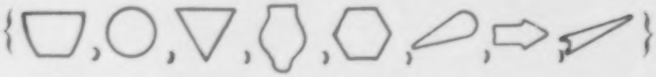
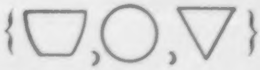
Figure 6-5b. 13 - 3 = 10.

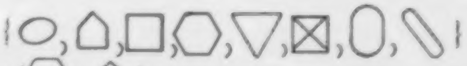
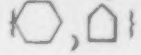
Figure 6-5. Two number lines showing 13 - (5 - 2) = 13 - 3 = 10.

Exercises - Chapter 6

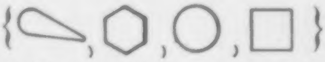
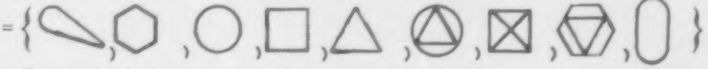
1. A = 
B = 
What is the number property of A ?
What is the number property of B ?
What is the number property of $A \cup B$?

2. A = 
List all the subsets of A.

3. A = 
B = 
Join to B a set C disjoint from B such that $B \cup C = A$.
This illustrates the set form of the __________ method.

4. If A = 
and B = 
exhibit A ~ B.

5. If from a set of 8 members we remove a set of 2 members, how many members does the resulting set have? This illustrates the ______ method.

6. If A = 
and C = 
exhibit B such that $A \cup B = C$. What is $N(B)$?

7. Show a representation on the number line which illustrates the fact that $10 - 3 = 7$. Use the same figure to illustrate the idea that $3 + 7 = 10$.

8. Show a representation on the number line which illustrates that the associative property does not hold under the operation of subtraction.

$$(9 - 6) - 3 \neq 9 - (6 - 3)$$

9. What operation is the inverse of adding 7 to any number? What is the inverse of subtracting 8 ?

*10. If A and B are disjoint, show that $(A \cup B) \sim B = A$. What happens if A and B are not disjoint?

11. Set $A = \{1,2,3,4,5,6,7,\ldots,100\}$. Which of the following sets are subsets of A ?
    a. $B = \{1,3,5,7,\ldots,99\}$
    b. $C = \{0,1,2,3\}$
    c. $D = \{\ \}$
    d. $E = \{4,8,12,16\}$
    e. $F = \{0,3,104\}$

Solutions for Problems

1. Choose A = {○,△,□,☆,◎,ℰ,▣} with N(A) = 7.
   Choose B = {☆,◎,□} which is a subset of A and N(B) = 3.
   A ~ B = {▣,○,ℰ,△}
   By definition, we know that 7 - 3 = N(A ~ B) = 4.

2. Choose A = {○,△,□,☆,◎,ℰ,▣} with N(A) = 7
   Choose $\overline{B}$ = { a, b, c } with $N(\overline{B})$ = 3
   Now choose a set C disjoint from both A and B
   C = {⊟, Ξ, ⍟, Φ} and N(C) = 4
   so that by matching $(\overline{B} \cup C)$ with A we can put $\overline{B} \cup C$ in one-to-one correspondence with A.

   $\overline{B} \cup C$ = { a, b, c, ⊟, Ξ, ⍟, Φ }
   A = {○,△,□,☆,◎,ℰ,▣}

   By definition we know that 7 - 3 = N(C) = 4

3. In the solution of Problem 1 we chose A = {○, △, □, ☆, ◎, ℰ, ▣} such that N(A) = 7.
In Problem 2 we chose the same A and for $\overline{B}$ we chose {a, b, c}. In Problem 2 matching of $\overline{B} \cup C$ with A the elements of $\overline{B}$ were matched with {○, △, □}. We can use this as the subset B which we need to use Definition 1. In this case A ~ B = {☆, ◎, ℰ, ▣} and this is the exact set which matches C, so N(A ~ B) = N(C) and the two methods must yield the same result since by ① a - b = N(A ~ B) and by ② a - b = N(C).

4. By using whole numbers 6, 4 we can illustrate the fact that a - b = n and b + n = a mean the same thing.
Thus 6 - 4 = 2 because 4 + 2 = 6
and 6 - 2 = 4 because 2 + 4 = 6 .

5. (a) inverse property of addition and subtraction
   (b) inverse property of addition and subtraction
   (c) inverse property of addition and subtraction showing grouping within the parentheses. 30 - 15 is another name for 15.
   (d) identity element of zero (Zero added to any number results in that number.)
   (e) identity element of zero (Zero subtracted from any number results in that number.)

6. (5 - 7) + 7 does not make sense in the present context because 5 - 7 is not a whole number. For any numbers a and b, (a - b) + b = a if $a \geq b$.

*7. To show that a + (b - a) = b if $b \geq a$ we use the commutative property of addition getting a + (b - a) = (b - a) + a, which by the third item in Properties of Subtraction is equal to b.

Chapter 7

ADDITION AND SUBTRACTION TECHNIQUES

Introduction

We have used sets to describe addition and subtraction and to develop its properties. Knowing that $5 + 3$ is the number of members in $A \cup B$, where A is a set of 5 members and B is a disjoint set of 3 members, enables us to count the members of $A \cup B$ and to discover that $5 + 3$ is 8. Knowing that $5 + 3 = 8$, from the definition of subtraction, we can see that $8 - 3 = 5$. This is fine, but it does not really help us much if we want to determine $892 + 367$ or $532 - 278$. To do problems like these quickly and accurately is a goal of real importance. It is a goal whose achievement is made much easier in our decimal system of numeration than in, for instance, the Chinese or Egyptian systems.

This unit is concerned with explaining the whys and wherefores of "carrying" and "borrowing" in the processes of computing sums and differences. Regrouping is a more accurate term for "carrying" and "borrowing" and will be used throughout this text.

We must recall how our system of numeration with base ten is built. What does the numeral 532 stand for? It stands for $500 + 30 + 2$; or 5 hundreds + 3 tens + 2 ones; or again, since one hundred stands for 10 tens, 532 stands for 5 groups of ten tens + 3 groups of ten + 2 ones. Also if we know that a number has 2 groups of ten tens and 7 groups of ten and 8 ones, we can write a numeral for that number in the form $(2 \times [10 \times 10]) + (7 \times 10) + (8 \times 1)$ or $200 + 70 + 8 = 278$. When we write the numeral in this stretched-out way, we have written it in "expanded form."

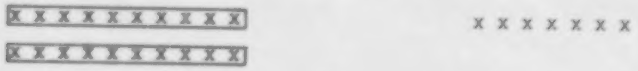
Regrouping Used in Addition

Let us assume that we know the addition facts for all the one-digit whole numbers and that we understand our decimal system of numeration. How does this help us? Let's try some examples. Suppose we want to add 42 and 37. Since we are adding (4 tens + 2 ones) to (3 tens + 7 ones) we get (7 tens + 9 ones) which we can write as 79.

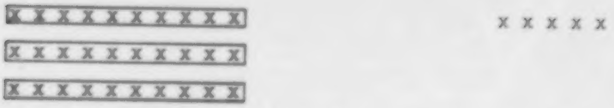
7

Essentially what we are doing is finding how many groups of tens and how many units we have and then using our system of numeration to write the correct numeral.

Let us add 27 and 35. This time we have (2 tens + 7 ones) + (3 tens + 5 ones) which may be illustrated:

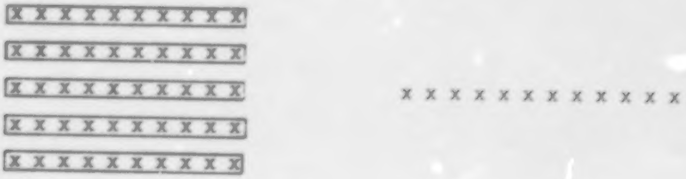


2 tens + 7 ones



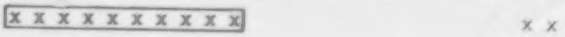
3 tens + 5 ones

By putting these groups together we now have:



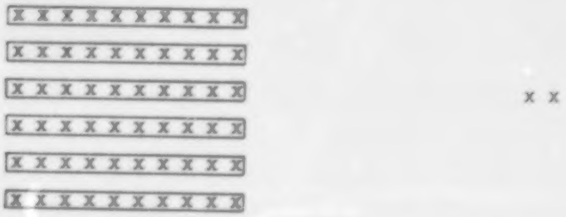
5 tens + 12 ones

We now regroup the 12 ones and get another set of 1 ten and 2 ones.



1 ten + 2 ones

We now add 5 tens + 1 ten + 2 ones.



5 tens + 1 ten + 2 ones

= 6 tens + 2 ones

= 62

When we add 68 and 57 we get 11 tens and 15 ones which we rewrite as follows:

$$\begin{array}{rl}
68 = 6 \text{ tens} + 8 \text{ ones} & \\
\underline{57 = 5 \text{ tens} + 7 \text{ ones}} & \\
11 \text{ tens} + 15 \text{ ones} & = (1 \text{ hundred} + 1 \text{ ten}) + (1 \text{ ten} + 5 \text{ ones}) \\
 & = 1 \text{ hundred} + 2 \text{ tens} + 5 \text{ ones} \\
 & = 125
\end{array}$$

If we switch from vertical to horizontal form, we write

$$\begin{aligned}
68 + 57 &= (60 + 8) + (50 + 7) \\
&= (60 + 50) + (8 + 7) && \text{use of associative property} \\
&= 110 + 15 && \text{and commutative property} \\
&= (100 + 10) + (10 + 5) \\
&= 100 + (10 + 10) + 5 && \text{use of associative property} \\
&= 100 + 20 + 5 \\
&= 125
\end{aligned}$$

Thus, for example, whenever we add 7 groups of one kind to 5 groups of the same kind, we get 12 groups which we write as 1 group of the next kind + 2 groups. It is this 1 which we can think of as "carrying" over to the next group.

Precisely the same process is used in adding three or more numbers. Once again the associative and commutative properties of addition are important. Thus:

$563 + 787 + 1384$ can be thought of as follows:

$$
\begin{aligned}
563 &= 500 + 60 + 3 = (5 \times 100) + (6 \times 10) + (3 \times 1) \\
787 &= 700 + 80 + 7 = (7 \times 100) + (8 \times 10) + (7 \times 1) \\
1384 &= 1000 + 300 + 80 + 4 = (1 \times 1000) + (3 \times 100) + (8 \times 10) + (4 \times 1)
\end{aligned}
$$

and the sum $563 + 787 + 1384$

$$
\begin{aligned}
&= (1 \times 1000) + (15 \times 100) + (22 \times 10) + (14 \times 1) \\
&= (1 \times 1000) + [(1 \times 1000) + (5 \times 100)] + [(2 \times 100) + \\
&\qquad (2 \times 10)] + [(1 \times 10) + (4 \times 1)] \\
&= [(1 \times 1000) + (1 \times 1000)] + [(5 \times 100) + (2 \times 100)] + \\
&\qquad [(2 \times 10) + (1 \times 10)] + (4 \times 1) \\
&= (2 \times 1000) + (7 \times 100) + (3 \times 10) + (4 \times 1) \\
&= 2000 + 700 + 30 + 4 \\
&= 2734
\end{aligned}
$$

This is usually abbreviated a great deal. But it is important that the underlying pattern be understood and the abbreviations recognized. Thus:

$$
\begin{array}{r}
500 + 60 + 3 \\
700 + 80 + 7 \\
1000 + 300 + 80 + 4 \\
\hline
1000 + 1500 + 220 + 14
\end{array}
$$

can be written with partial sums indicated as:

$$
\begin{array}{rl}
563 & \\
787 & \\
1384 & \\
\hline
14 & \text{sums of ones} \\
220 & \text{sums of tens} \\
1500 & \text{sums of hundreds} \\
1000 & \text{sums of thousands} \\
\hline
2734 &
\end{array}
$$

and the operation is still further abbreviated to:

$$
\begin{array}{r}
①②① \\
563 \\
787 \\
1[illegible] \\
\hline
2734
\end{array}
$$

Finally, by omitting even the carry over numerals we get:

$$
\begin{array}{r}
563 \\
787 \\
1384 \\
\hline
2734
\end{array}
$$

Regrouping Used in Subtraction

In the operation of addition we operate upon two numbers called addends to produce a unique third number called the sum. The process of subtraction can be thought of as finding an unknown addend which added to a known addend will produce a known sum. If n is the unknown addend, $a - b = n$ is the number n such that $a = b + n$. This is the third definition of subtraction discussed in Chapter 6. When the numbers are small, the unknown addend is easily determined, if the usual addition facts are known. Thus $6 - 2 = 4$ because $6 = 2 + 4$ and $17 - 9 = 8$ because $17 = 9 + 8$. We easily do $49 - 17$ as this is simply the answer to the question: what number added to 1 ten and 7 ones will produce 4 tens and 9 ones? It is, of course, 3 tens and 2 ones which we write directly as 32.

An example with larger numbers such as $523 - 376$ will perhaps make clearer the usefulness of expressing a number in expanded form in order to understand the process of subtraction. We are focusing for the moment on the conceptual aspects of subtraction. Of course, we do not ordinarily use this form in computation. Let us go back to our example, $523 - 376$; that is, find the number which must be added to 376 to yield the sum 523. Since recognition of the answer is not immediate, we proceed to something like this. First we write

$$\begin{array}{r} 523 \\ -\ \underline{376} \end{array}$$

Now the usefulness of expressing 523 and 376 in forms that clearly delineate what each place means is apparent. The objective is to put at least as great a number of ones in the ones' place in the sum 523 as in the ones' place in the known addend 376; to put at least as many tens in the tens' place in the sum as in the tens' place in the known addend; and so on.

Thus we may write:

$$\begin{aligned} 523 &= 500 + 20 + 3 \\ 376 &= 300 + 70 + 6 \end{aligned}$$

as an initial step in accomplishing our purpose.

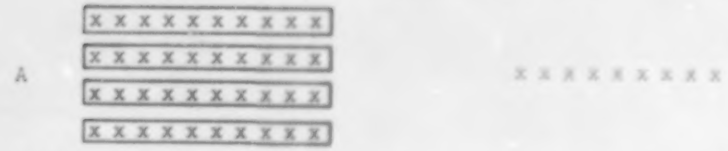
Next we write

$$\begin{aligned} 523 &= 500 + 20 + 3 = 500 + 10 + 13 = 400 + 110 + 13 \\ 376 &= 300 + 70 + 6 = 300 + 70 + \ \ 6 = \underline{300 + \ \ 70 + \ \ 6} \end{aligned}$$

and the answer can be seen to be: $100 + 40 + 7$ or 147.

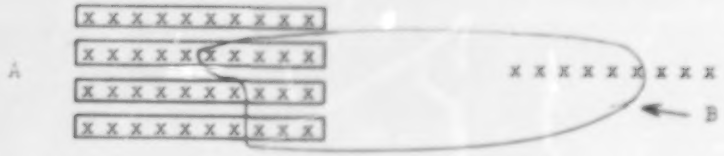
There is no need for a special term to describe what we are doing and we do not usually write it out in detail. The primary issue is to recognize the need for at least as many ones in the ones' place of the sum as in the ones' place of the addend, at least as many tens in the tens' place of the sum as in the tens' place of the addend, etc. Of course, this is the process usually known as "regrouping."

Let us now return to the subtraction 49 - 17, and let us reexamine this problem from the point of view of the first definition of subtraction in Chapter 6, that is, in terms of remainder sets. We can take for our set A a collection of 49 x's arranged as follows:



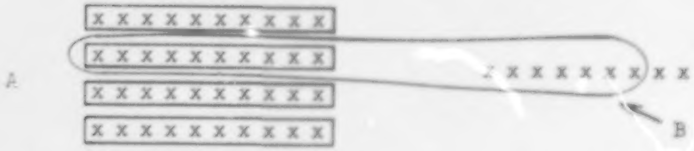
Now we need to pick a subset B of A which contains 17 members. Then the number of members of the remainder set A - B will be 49 - 17.

There are many ways to choose B . One of them is this:



But when we choose B this way, the remainder set A - B is not easy to count. Some of the original bundles of ten have been broken up, and only pieces of them are in A - B.

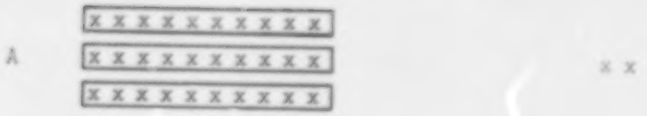
It is much better if we choose B so as to either include all of a bundle of ten or none of it. Here is one way:



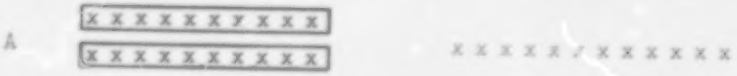
Now it is easy to count the remainder set A - B . It can be done in two steps. Looking at the right hand side above, we see that the number of units in the remainder set is 9 - 7 = 2. Looking at the left hand side above, we see that the number of bundles of ten in the remainder set is 4 - 1 = 3. Therefore the number of members in the remainder set is 32.

The important thing to notice is that since we dealt only with complete bundles of ten, we could count these using only "small" numbers.

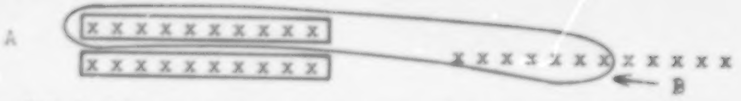
Now, let us examine in the same way another problem: 32 - 17. We can pick A to be a set of 32 x's :


A
x x x x x x x x x x
x x x x x x x x x x
x x x x x x x x x x
x x

We need to pick a subset B with 17 members, that is, one bundle of ten and seven units. But A has only two units, so we will have to use some of the members of A in the bundles of ten. As we saw above, it is best if we use only whole bundles. Therefore, we will take one of the bundles of ten in A and put it in with the units. Now A looks like this:


A
x x x x x x x x x x
x x x x x x x x x x
x x x x x x x x x x x x

Now it is easy to see how we can pick a convenient subset B which has 17 members. Here is one:


A
x x x x x x x x x x
x x x x x x x x x x
x x x x x x x x x x x x
B

Now it is easy to count the remainder set A - B. The number of units is 12 - 7 = 5 and the number of tens is 2 - 1 = 1. Therefore 32 - 17 = 15.

Let us repeat this problem, 32 - 17, using expanded form rather than sets. We can write 32 as:

3 tens + 2 ones
or 2 tens + 1 ten + 2 ones
or 2 tens + 12 ones

Written another way:

$$32 = 30 + 2 = (20 + 10) + 2 = 20 + (10 + 2) = 20 + 12$$

Now the number to be added to 17 (1 ten and 7 ones) to give 32 (2 tens and 12 ones) is seen to be 1 ten and 5 ones, or 15. This result can also be readily seen either by thinking in terms of a remainder set or in terms of a missing addend.

Figure 7-1 displays the computation $68 - 49$ in some detail using the procedures discussed here.

Horizontal Form	Vertical Form
Step 1: 68 and 49 are written in expanded form	
$68 - 49 = (60 + 8) - (40 + 9)$	$\begin{array}{l} 68 = (60 + 8) \\ \underline{49 = (40 + 9)} \end{array}$
Step 2: 68 is regrouped as $50 + 18$ because, looking ahead, we see the need for more ones in the ones' place because $(8 - 9)$ cannot be computed with whole numbers.	
$68 - 49 = (50 + 18) - (40 + 9)$	$\begin{array}{r} 50 + 18 \\ \underline{40 + 9} \end{array}$
Step 3: Now we rearrange so as to subtract 9 from 18 and 40 from 50.	
$\begin{aligned} 68 - 49 &= (50 - 40) + (18 - 9) \\ &= \quad 10 \quad + \quad 9 \\ &= 19 \end{aligned}$	$\begin{array}{r} 50 + 18 \\ \underline{40 + 9} \\ 10 + 9 \end{array} = 19$

Figure 7-1. The subtraction: $68 - 49$.

It is recognized that this explanation is long and wordy in written exposition. The actual computation is fairly brief. Details are supplied in an attempt to explain the basis for regrouping in computational processes which are frequently executed properly but almost universally misunderstood.

Another Property of Subtraction

In Chapter 6 we considered some of the important properties of subtraction. In the example worked in Figure 7-1, we used another important property which now is stated formally:

> If $a + b$ is the name of one number and $c + d$ is the name of a second number, and if $a + b \geq c + d$ and also $a \geq c$ and $b \geq d$ then
> $(a + b) - (c + d) = (a - c) + (b - d)$.

Figure 7-2 gives another illustration of this property for $68 - 42$. It shows that in writing a subtraction in the vertical form the property is applied automatically.

Horizontal Form

$$\begin{aligned} 68 - 42 &= (60 + 8) - (40 + 2) \\ &= (60 - 40) + (8 - 2) \\ &= 20 + 6 \\ &= 26 \end{aligned}$$

Vertical Form

$$\begin{array}{r} 68 \\ -\ \underline{42} \\ 26 \end{array}$$

Figure 7-2. Example of the rule $(a + b) - (c + d) = (a - c) + (b - d)$

The relation between the illustration and the statement of the property is seen if you think of a replaced by 60, b replaced by 8, c replaced by 40 and d replaced by 2.

The property is applicable to other subtractions such as 342 - 187. More steps are required, however, because of the necessary regrouping.

Horizontal Form

$$\begin{aligned} 342 - 187 &= (300 + 40 + 2) - (100 + 80 + 7) \\ &= (300 + 30 + 12) - (100 + 80 + 7) \\ &= (200 + 130 + 12) - (100 + 80 + 7) \\ &= (200 - 100) + (130 - 80) + (12 - 7) \\ &= 100 + 50 + 5 \\ &= 155 \end{aligned}$$

Vertical Form

$$\begin{array}{l} 342 = 300 + 40 + 2 = 300 + 30 + 12 = 200 + 130 + 12 \\ 187 = 100 + 80 + 7 = 100 + 80 + \ \ 7 = \underline{100 + \ \ 80 + \ \ 7} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\ 100 + \ \ 50 + \ \ 5 \ \ = \ \ 155 \end{array}$$

Summary

It is important that students thoroughly understand the positional system and think of regrouping in many different ways. This understanding of the positional system and its many regrouping potentials leads the student to see that each numeral indicates a sum of parts and this is useful in explaining the techniques used in addition and subtraction.

Exercises - Chapter 7

1. Use expanded notation to do the following sums.

a. 246
139

b. 784
926

c. 777
964

d. 123
987

e. 486
766

f. 949
892

2. How would you regroup the 300 to do the following subtraction example?

300
178

Show the answer in various groupings.

3. Which of the following are other names for 8000 ?

a. 8000 ones
b. 8021 - 21
c. 8 thousands
d. 7000 + 1000
e. 800 tens
f. 80 hundreds
g. 8000 - 0
h. 8000 + 0

4. What property or properties of addition are you using when you check addition by adding from top to bottom after you have added from bottom to top?

a. when you use two addends.
b. when you use three or more addends.

5. Do the following subtraction examples both horizontally and vertically showing all the regrouping.

a. 764
199

b. 402
139

c. 710
287

Chapter 8

MULTIPLICATION

Introduction

Chapters 5, 6 and 7 dealt with addition and subtraction of whole numbers. The point was made but is worth repeating and emphasizing here, that addition is an operation on two numbers which yields a third number. Thus, given the numbers 7 and 3, addition yields 10. The given numbers are not necessarily different numbers; nor is it meant that the third number referred to is necessarily different from any of the given numbers. For example,

for the given numbers 4 and 4, addition yields 8;
for the given numbers 4 and 0, addition yields 4; and
for the given numbers 0 and 0, addition yields 0.

All that is meant is that if a is a whole number and b is a whole number, then $a + b$ is always a whole number. Thus, the set of whole numbers is said to be closed under addition; in other words, addition has the property of closure for whole numbers.

On the other hand, if a is a whole number and b is a whole number, it is not always true that $a - b$ is a whole number. If a is 5 and b is 8 for example, $5 - 8$ is not a whole number. Recall that by one of the definitions for subtraction, $5 - 8$ is that number which added to 8 is 5. That is,

$$5 - 8 = n \text{ if and only if } 8 + n = 5,$$

and since neither of these statements is true for any whole number n, it can be concluded that $5 - 8$ is not a whole number.

Problem *

1. For each of the following pairs of numbers (a, b), tell whether $a - b$ is a whole number and whether $b - a$ is a whole number. For example, for (5, 8), $a - b$ is not a whole number, but $b - a$ is a whole number.

   a. (4, 9) b. (9, 7) c. (0, 3) d. (0, 0)

Another interpretation for subtraction was presented as removing a subset from a set; it is impossible to remove a subset of 8 from a set of 5. Hence again, $5 - 8$ does not name a whole number. Many such

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* Solutions for problems in this chapter are on page 90.

exceptions can be found, but it requires only one exception to show that the set of whole numbers is not closed under subtraction.

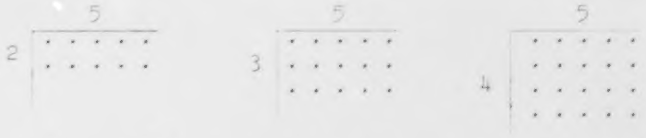
Addition was mentioned as a special way of assigning a unique number to an ordered pair of numbers. For example,

to the ordered pair $(7, 3)$ is assigned
the number 10; that is, $7 + 3 = 10$.

Since addition is an operation on a pair of numbers, addition is said to be a binary operation. (The "bi" is the same "bi" as in "bicycle" or "biped" or "bivalve." In each case it indicates "two.") There are many possible binary operations on an ordered pair of numbers; the most familiar binary operations are addition, subtraction, multiplication and division. In each case, a pair of numbers is combined to produce a third number.

Multiplication as such a binary operation is the subject of this chapter. Multiplication assigns a unique number called the product to a pair of numbers. When the number 20 is associated with the ordered pair $(4, 5)$, multiplication may be indicated. Numbers were defined in terms of sets of objects, and an interpretation of $4 \times 5 = 20$ may be referred to the notion of sets. How 20 is determined from the two numbers 4 and 5 is central to the interpretation.

For a physical interpretation of 4×5 we may set up a rectangular array of 4 rows with 5 objects in each row and count the number of objects in the array. An array for 4×5 and arrays for various other such products are illustrated in Figure 8-1.



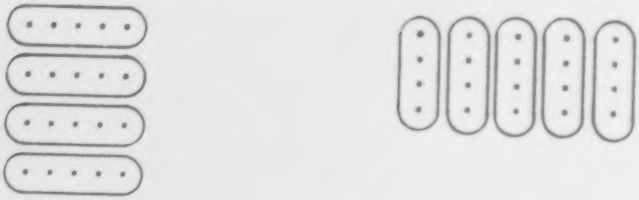
$2 \times 5 = 10$ $3 \times 5 = 15$ $4 \times 5 = 20$

Figure 8-1. Various arrays to illustrate multiplication.

From Figure 8-2a it may be seen that a 4 by 5 array is the union of 4 disjoint sets, each set having 5 members. Consequently, 4×5 can be computed by the successive addition

$$\overbrace{5 + 5 + 5 + 5}^{4 \text{ addends}},$$

that is, 5 is used as an addend 4 times. (This is sometimes referred to as the repeated addition description of multiplication.) It is also true that the array is the union of 5 disjoint sets, each set having 4 members



4 sets, 5 members
in each set.

(a)

5 sets, 4 members
in each set.

(b)

Figure 8-2.

(see Figure 8-2b). Thus 4 × 5 can be computed by the successive addition of 4 as an addend 5 times.

Definition of Multiplication

The result of multiplying two numbers is their *product* and may be defined in terms of counting sets as follows:

> Given numbers *a* and *b*, an *a* by *b* rectangular array of objects can be constructed such that there are *a* rows and *b* columns in the array. The product, written a × b and read *a* times *b*, is defined to be the number of objects in the array.

This definition gives a means of computing a product by physical manipulation of sets, and makes several of the important properties of multiplication fairly evident.

As in the special case of the 4 × 5 array, in general an *a* by *b* array is the union of *a* disjoint sets (rows), each having *b* members. Hence, a × b can be computed by the successive addition of *b* to itself *a* times. It is also true that the array is the union of *b* disjoint sets (columns) each of which has *a* members. If the first number indicates the number of rows and the second the number of columns, then a × b is an array of *a* rows and *b* columns. The *a* and *b* are called *factors* and a × b is the *product*.

The product of every pair of whole numbers is a whole number; hence, the set of whole numbers is *closed* under multiplication.

Problems

2. For each pair (a, b) where a is thought of as the number of rows and b the number of columns, draw an array to show the product, a × b.

   a. (2, 5)
   b. (5, 2)
   c. (5, 6)
   d. (6, 5)
   e. (1, 4)
   f. (4, 1)
   g. (0, 3)
   h. (7, 0)

3. For the number pair (a, b) where a = 3 and b = 3, draw an array to show the product, a × b; also draw an array to show b × a.

4. In adding, there is a particular number a such that a + a = a; find this number a.

5. In multiplication, is there a number a such that a × a = a?

6. Are there more than one number a such that a × a = a?

7. If possible, draw an array for a × a such that a × a = a.

The rectangular array offers a natural way of displaying various results of mixing and matching. For example, if a store offers a sweater and skirt ensemble with each sweater available in white, blue or gray, and each skirt available in white, blue, gray or red, the various ensembles possible may be displayed as follows:

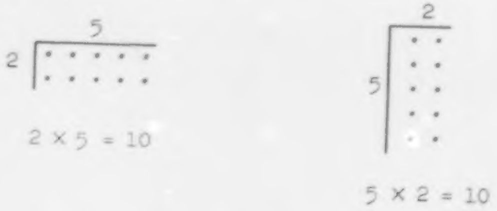
	white skirt (w)	gray skirt (g)	red skirt (r)	blue skirt (b)
white sweater (W)	W w	W g	W r	W b
blue sweater (B)	B w	B g	B r	B b
gray sweater ([illegible])	G w	G g	G r	G b

Problem

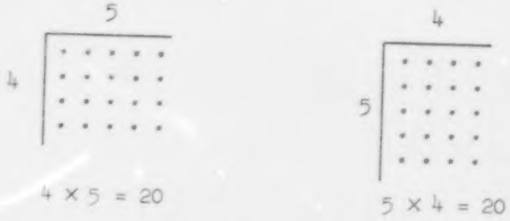
8. Doris has 5 blouses and 6 skirts. How many mix and match outfits can Doris make? Assume that each blouse will match with each skirt.

The Commutative Property of Multiplication

From the study of various arrays another property of whole numbers may be discovered.



Notice that $2 \times 5 = 5 \times 2$.



Notice that $4 \times 5 = 5 \times 4$.

Figure 8-3. Models to illustrate the commutative property of multiplication.

In general, if a and b are whole numbers, $a \times b = b \times a$. This is the Commutative Property of Multiplication. This property may be seen from the definition directly, for an a by b array can be changed into a b by a array simply by rotating through 90 degrees.

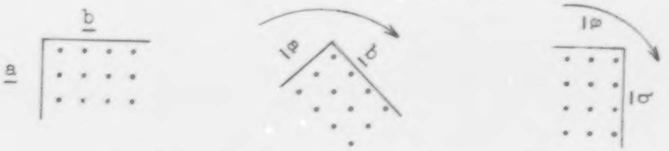


Figure 8-4. The commutative property illustrated by same model.

Commuting the order of factors does not alter the product $(3 \times 7 = 7 \times 3)$. Commutativity is pedagogically quite important because, for example, the product of 3 and 7 seems simpler than the product of 7 and 3. This property also reduces the number of multiplication facts to be remembered since $a \times b$ and $b \times a$ may be learned simultaneously, and the property is useful in simplifying calculations.

Problem

9. Show by trying to indicate the steps in repeated addition how the commutative property of multiplication would simplify the calculation of 1000×3.

The Associative Property of Multiplication

Multiplication is a binary operation, and given three numbers, (a, b, c), it is not immediately obvious whether any meaning may be attached to $a \times b \times c$. One may note that if a and b are whole numbers, $(a \times b)$ is a single whole number. Now this whole number may be paired with c to obtain a product. This is indicated by $(a \times b) \times c$. Or alternately, one may note that $(b \times c)$ is a whole number, and may be paired with a to obtain a product which may be described as $a \times (b \times c)$. For example,

$$(3 \times 2) \times 4 = 6 \times 4 = 24,$$
$$\text{and}\quad 3 \times (2 \times 4) = 3 \times 8 = 24.$$

In general, it is true that $(a \times b) \times c = a \times (b \times c)$, so it does not matter how the pairings are grouped; a unique number is assigned to the triple (a, b, c) as the product. This freedom of grouping is similar to the associative property of addition, and is called the Associative Property of Multiplication. That is, for any three whole numbers a, b, c,

$$(a \times b) \times c = a \times (b \times c).$$

This is, in fact, the property which permits $a \times b \times c$ to be written without parentheses. The same procedure used above also permits multiplication to be extended to more than three factors.

Problem

10. Show by grouping with parentheses how $a \times b \times c \times d$ may be regarded as a product involving 3 factors instead of 4 for each of the following:
    a. $2 \times 3 \times 4 \times 5 = 2 \times 3 \times 20$
    b. $2 \times 3 \times 4 \times 5 = 6 \times 4 \times 5$
    c. $2 \times 3 \times 4 \times 5 = 2 \times 12 \times 5$

Since multiplication has the commutative and the associative properties, multiplication has the same flexibility in grouping and rearranging of the factors as addition has in the grouping and rearranging of the addends. For example, for

> $25 \times 15 \times 3 \times 4$, it can be seen that the product of 25 and 4 is 100 and the product of 3 and 15 is 45, consequently, $25 \times 15 \times 3 \times 4 = 4500$.

This flexibility amounts to a sort of "do-it-whichever-way-we-want" principle in any problem where only multiplications are involved.

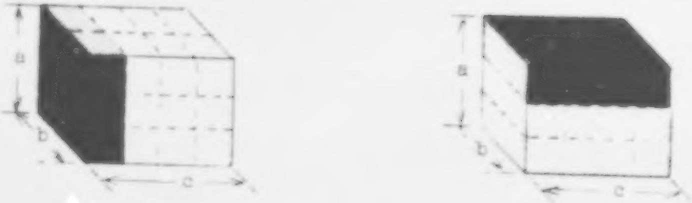
Problems

11. Show that $2 \times 3 \times 4 = 8 \times 3$ involves both the commutative and the associative properties of multiplication.

12. What property or properties are involved in each of the following?

   a. $2 \times 3 \times 4 = 2 \times 12$
   b. $2 \times 3 \times 4 = 3 \times 8$
   c. $2 \times 3 \times 4 = 6 \times 4$
   d. $2 \times 3 \times 4 = 2 \times 4 \times 3$
   e. $2 \times 3 \times 4 = 3 \times 2 \times 4$
   f. $4 \times 3 \times 2 = 4 \times 3 \times 2$

The physical model of a box made up of cubical blocks with dimensions a by b by c, may be used to illustrate the associativity of multiplication.



a × b blocks in each vertical slice; c vertical slices.

b × c blocks in each horizontal slice; a horizontal slices.

Figure 8-5. Model illustrating the associative property of multiplication.

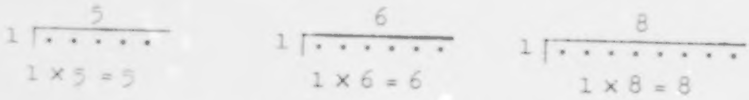
The number of blocks in such a box is $(a \times b) \times c$ and is also $a \times (b \times c)$ indicating that it is true that $(a \times b) \times c = a \times (b \times c)$.

The Identity for Multiplication

The number 1 occupies, with respect to multiplication, the same position that 0 occupies with respect to addition. Notice that,

$$1 \times 3 = 3 \times 1 = 3,$$
$$1 \times 5 = 5 \times 1 = 5,$$
$$1 \times 6 = 6 \times 1 = 6,$$
$$1 \times 8 = 8 \times 1 = 8.$$

It is true that $1 \times n = n$ for all numbers n because a 1 by n array consists of only one row having n members, and therefore the entire array contains exactly n members. Since $1 \times n = n$, the number 1 is called the identity element for multiplication.



$1 \times 5 = 5$ $\quad$ $1 \times 6 = 6$ $\quad$ $1 \times 8 = 8$

Figure 8-6. Arrays of 1 by n.

Multiplication Property of 0

The number 0, besides playing the role of the identity element for *addition*, also has a rather special property with respect to *multiplication*. The number of members in a 0 by n array (that is, an array with 0 rows, each having n members) is 0 because the set of members of this array is empty. Similarly, the set of members of an array of n rows, each of them having 0 members, is empty. Thus for any number n,

$$0 \times n = n \times 0 = 0.$$

What has been done so far shows that multiplication, as with addition, is an operation on the whole numbers which has the properties of closure, commutativity and associativity. There is a special number 1 that is an identity for multiplication just as 0 is an identity for addition. Moreover, 0 plays a special role in multiplication for which there is no corresponding property in addition.

The Distributive Property

We have seen that multiplication may be described by repeated addition. Aside from this, there is another important property that links the two operations. This property which we shall now study is the basis, for example, for the following statement:

$$4 \times (7 + 2) = (4 \times 7) + (4 \times 2).$$

This example may be verified by noting that both $4 \times (7 + 2)$ and $(4 \times 7) + (4 \times 2)$ give the same result:

$$4 \times (7 + 2) = 4 \times 9 = 36, \text{ and}$$
$$(4 \times 7) + (4 \times 2) = 28 + 8 = 36.$$

The property is called the *Distributive Property of Multiplication over Addition*. The distributive property states that if a, b and c are any whole numbers, then

$$a \times (b + c) = (a \times b) + (a \times c).$$

The distributive property may be illustrated by considering an a by $b + c$ array.

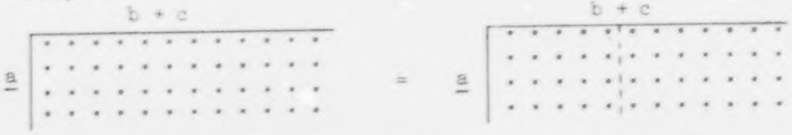
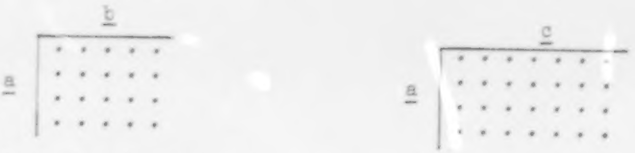


Figure 8-7. An a by $b + c$ array.

It is true that this array is formed from an a by b array and an a by c array.



An a by b array　　An a by c array

Figure 8-8. Makeup of an a by $b + c$ array.

Consequently, the number $a \times (b + c)$ of members in the large array is the sum of $(a \times b)$ and $(a \times c)$, the numbers of members of the subsets. That is, $a \times (b + c) = (a \times b) + (a \times c)$.

Since multiplication is commutative, both the "left hand" and the "right hand" distributive properties hold, that is,

Left hand: $a \times (b + c) = (a \times b) + (a \times c)$, and
Right hand: $(b + c) \times a = (b \times a) + (c \times a)$.

For example, by these distributive properties,

Left hand: $3 \times (5 + 8) = (3 \times 5) + (3 \times 8)$, and
Right hand: $(4 + 7) \times 2 = (4 \times 2) + (7 \times 2)$.

Recalling that when we say $A = B$ we mean A and B both name the same thing, then if $A = B$, it really makes no difference whether we write $A = B$ or $B = A$. With this in mind, since the left hand distributive property says that $a \times (b + c)$ and $(a \times b) + (a \times c)$ name the same number, the statement

$$a \times (b + c) = (a \times b) + (a \times c)$$

can equally well be written as

$$(a \times b) + (a \times c) = a \times (b + c).$$

For example,

$$(3 \times 5) + (3 \times 8) = 3 \times (5 + 8).$$

Similarly, the right hand distributive property may be expressed as either

$$(b + c) \times a = (b \times a) + (c \times a)$$

or

$$(b \times a) + (c \times a) = (b + c) \times a.$$

For example,

$$(4 \times 2) + (7 \times 2) = (4 + 7) \times 2.$$

The distributive property is very important as it is the basis of shortcutting many a computation that appears involved. Thus,

$$\begin{aligned}\text{Left hand:}\quad (5 \times 4) + (5 \times 6) &= 5 \times (4 + 6)\\ &= 5 \times 10 = 50;\quad \text{also}\\ \text{Right hand:}\quad (7 \times 9) + (3 \times 9) &= (7 + 3) \times 9\\ &= 10 \times 9 = 90.\end{aligned}$$

The convenience in such shortcutting may be further illustrated by the following examples:

$$\begin{aligned}&(9 \times 17) + (9 \times 83) = 9 \times (17 + 83) = 9 \times 100 = 900;\\ &(24 \times 17) + (26 \times 17) = (24 + 26) \times 17 = 50 \times 17 = 850;\\ &(854 \times 673) + (146 \times 673) = (854 + 146) \times 673 = 1000 \times 673 = 673{,}000;\\ &(84 \times 367) + (84 \times 633) = 84 \times 1000 = 84{,}000.\end{aligned}$$

Problems

13. Use the distributive property to compute each of the following:
    a. $(57 \times 7) + (57 \times 93)$
    b. $(57 \times 8) + (57 \times 93)$ [Hint: $8 = 7 + 1$]

14. Show that $(57 \times 5) + (57 \times 5) = 57 \times 10$ by the distributive property.

The question may now arise: Given three numbers 4, 7, 2, is $(4 + 7) \times 2$ the same as $4 + (7 \times 2)$? Is $4 \times (7 + 2)$ the same as $(4 \times 7) + 2$? Neither of these is true since:

$$\begin{aligned}(4 + 7) \times 2 &= 11 \times 2 = 22,\\ 4 + (7 \times 2) &= 4 + 14 = 18,\\ 4 \times (7 + 2) &= 4 \times 9 = 36,\\ (4 \times 7) + 2 &= 28 + 2 = 30.\end{aligned}$$

Hence, extra caution must be exercised when both addition and multiplication are intermixed.

One might question whether addition distributes over multiplication. That is, is it always the case that

$$a + (b \times c) = (a + b) \times (a + c)?$$

This would be false if any set of numbers a, b and c can be found that would disprove the statement. For example, $a = 1$, $b = 3$, and $c = 2$ may be tried. For these values,

$$\begin{aligned}&a + (b \times c) = 1 + (3 \times 2) = 1 + 6 = 7,\ \text{but}\\ &(a + b) \times (a + c) = (1 + 3) \times (1 + 2) = 4 \times 3 = 12.\end{aligned}$$

So it cannot be stated that $a + (b \times c)$ is always equal to $(a + b) \times (a + c)$.

Another related question may arise as to whether multiplication distributes over subtraction; that is, whether it is true that

$$a \times (b - c) = (a \times b) - (a \times c).$$

For the values $a = 1$, $b = 3$, $c = 2$,

$$a \times (b - c) = 1 \times (3 - 2) = 1 \times 1 = 1, \text{ and}$$
$$(a \times b) - (a \times c) = (1 \times 3) - (1 \times 2) = 3 - 2 = 1.$$

Other examples may be tried and it will turn out that in every instance multiplication does distribute over subtraction, subject to the restriction of course that $b \geq c$, otherwise $b - c$ is not defined for whole numbers.

As a further remark that may be of interest, consider that 3×5 can be thought of as 3 fives. Since $5 = 1 + 1 + 1 + 1 + 1$,

$$\begin{aligned} 3 \text{ fives} &= 3 \times 5 = 3 \times (1 + 1 + 1 + 1 + 1) \\ &= 3 + 3 + 3 + 3 + 3 = 5 \text{ threes} \\ &= 5 \times 3. \end{aligned}$$

So by the distributive property, the commutative property of multiplication may again be illustrated.

Summary

The properties of addition and multiplication developed so far for whole numbers may be summarized as follows, where a, b and c are whole numbers.

1. Whole numbers are CLOSED under addition and multiplication
   $a + b$ and $a \times b$ are whole numbers.
2. Addition and multiplication are COMMUTATIVE operations
   $a + b = b + a$ and $a \times b = b \times a$.
3. Addition and multiplication are ASSOCIATIVE operations
   $(a + b) + c = a + (b + c)$ and $(a \times b) \times c = a \times (b \times c)$.
4. There is an IDENTITY element 0 for addition and an IDENTITY element 1 for multiplication
   $a + 0 = a$ and $a \times 1 = a$.
5. Multiplication is DISTRIBUTIVE over addition
   $a \times (b + c) = (a \times b) + (a \times c)$.

6. Multiplication is DISTRIBUTIVE over subtraction whenever subtraction is defined

   $a \times (b - c) = (a \times b) - (a \times c)$.

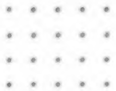
7. Zero has a special multiplication property

   $0 \times a = 0$.

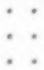
Exercises - Chapter 8

1. What mathematical sentence is suggested by each of the arrays below?

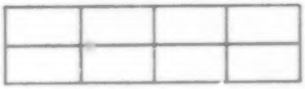
   a.



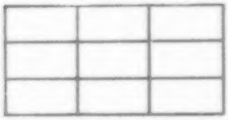
   b.



   c.



   d.



2. Mr. Rhodes is buying a two-tone car. The company offers tops in 5 colors and bodies in 3 colors. Draw an array that shows the various possible results, assuming that none of the body colors are the same as any of the top colors.

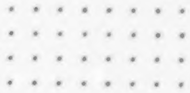
3. Mr. Rhodes is buying a two-tone car. Colors available for the top are: red, orange, yellow, green and blue. Colors available for the body are: red, yellow and blue. Draw an array to show the various possible results. If Mr. Rhodes insists that the car must be two-toned, how many choices does he have?

*4. In an experiment, 3 varieties of plants are each tested with 5 different plant hormones. A plot is reserved for each variety with each hormone. How many plots does this experiment require?

*5. An ensemble of sweater and skirt is offered with the sweater available in five different colors and the skirt in 4 colors. The skirt also comes in either straight or flare style for each of the 4 colors. How many different ensembles are possible?

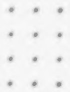
6. Complete the following statements by using either "always," "not always," or "never."
   a. The product of two even numbers is ________ an even number.
   b. The product of two odd numbers is ________ an even number.
   c. The product of two even numbers is ________ an odd number.
   d. The product of an odd number and an even number is ________ an even number.

7. Here is an array separated into two smaller arrays.



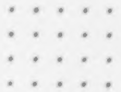
$(n = 4 \times 8)$

Array A



$(p = 4 \times 3)$

Array B



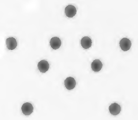
$(q = 4 \times 5)$

Array C

   a. How many dots are in array A? array B? array C?
   b. Does $n = p + q$?
   c. Does $4 \times 8 = (4 \times 3) + (4 \times 5)$?

8. Show two different ways of working the following problem to illustrate the distributive property.
   "Jerry sold paperbacks at 90 cents each. On Monday, he sold 23 and on Tuesday, he sold 27. How much money did he collect?

9. A familiar puzzle problem calls for planting 10 trees in an orchard so there are 5 rows with 4 trees in each row. The solution is in the form of the star shown in the figure to the right. Why doesn't this star illustrate the product of 5 and 4?



10. The middle section of an auditorium seats 28 to a row, and each side section seats 11 to a row. What is the capacity of this auditorium if there are 20 such rows?

11. What property of numbers is used in the following regrouping?
    $96 + 248 = 96 + 4 + 244 = 100 + 244 = 344$.

12. Use the do-it-whichever-way-we-want principle to get the answer quickly.
    a. $5 \times 4 \times 3 \times 2 \times 1$
    b. $125 \times 7 \times 3 \times 8$
    c. $250 \times 14 \times 4 \times 2$

13. Which of the following are true?
    a. 3 + (4 × 2) = (3 + 4) × (3 + 2)
    b. 3 × (4 - 2) = (3 × 4) - (3 × 2)
    c. (4 + 6) × 2 = (4 × 2) + (6 × 2)
    d. (4 + 6) + 2 = (4 + 2) + (6 + 2)
    e. 3 + (4 × 2) = (3 × 4) + (3 × 2)

14. Make each of the following a true statement illustrating the distributive property.
    a. 3 × (4 + ___) = (3 × 4) + (3 × 3)
    b. 2 × (___ + 5) = (2 × 4) + (___ × 5)
    c. 13 × (6 + 4) = (13 × ___) + (13 × ___)
    d. (2 × 7) + (3 × ___) = (___ ___) × 7

Solutions for Problems

1. a. a - b = 4 - 9 is not a whole number.
       b - a = 9 - 4 is a whole number.

    b. a - b = 9 - 7 is a whole number.
       b - a = 7 - 9 is not a whole number.

    c. a - b = 0 - 3 is not a whole number.
       b - a = 3 - 0 is a whole number.

    d. a - b = 0 - 0 is a whole number.
       b - a = 0 - 0 is a whole number.

```
2.  a.    . . . . .                  e.    . . . .
          . . . . .

    b.     . .                       f.    .
           . .                             .
           . .                             .
           . .                             .
           . .

    c.  . . . . . .                  g.  (the empty set)
        . . . . . .
        . . . . . .
        . . . . . .
        . . . . . .

    d.  . . . . .                    h.  (the empty set)
        . . . . .
        . . . . .
        . . . . .
        . . . . .
        . . . . .
```

3. a × b   b × a

   . . .   . . .
   . . .   . . .
   . . .   . . .

4. 0

5. Yes

6. Yes; both $0 \times 0 = 0$ and $1 \times 1 = 1$, so a may be either 0 or 1.

7. · (i.e., a 1×1 array)

8. $5 \times 6 = 30$

9. $1000 \times 3 = \overbrace{3 + 3 + \dots + 3}^{\text{1000 addends}} = 3000$

   $3 \times 1000 = 1000 + 1000 + 1000 = 3000.$

10. a. $2 \times 3 \times 4 \times 5 = 2 \times 3 \times (4 \times 5) = 2 \times 3 \times 20$

    b. $2 \times 3 \times 4 \times 5 = (2 \times 3) \times 4 \times 5 = 6 \times 4 \times 5$

    c. $2 \times 3 \times 4 \times 5 = 2 \times (3 \times 4) \times 5 = 2 \times 12 \times 5$

11. $2 \times 3 \times 4 = 2 \times 4 \times 3$, commutative

    $= (2 \times 4) \times 3$, associative

    $= 8 \times 3$, renaming

12. a. $2 \times 3 \times 4 = 2 \times (3 \times 4) = 2 \times 12$, associative

    b. $2 \times 3 \times 4 = 3 \times 2 \times 4$, commutative

    $= 3 \times (2 \times 4)$, associative

    $= 3 \times 8$

    c. $2 \times 3 \times 4 = (2 \times 3) \times 4 = 6 \times 4$, associative

    d. $2 \times 3 \times 4 = 2 \times 4 \times 3$, commutative

    e. $2 \times 3 \times 4 = 3 \times 2 \times 4$, commutative

    f. $4 \times 3 \times 2 = 4 \times 3 \times 2$, none involved

13. a. $(57 \times 7) + (57 \times 93) = 57 \times (7 + 93)$

    $= 57 \times 100 = 5700$

    b. $(57 \times 8) + (57 \times 93) = (57 \times [1 + 7]) + (57 \times 93)$

    $= (57 \times 1) + (57 \times 7) + (57 \times 93)$

    $= (57 \times 1) + (57 \times [7 + 93])$

    $= (57 \times 1) + (57 \times 100)$

    $= 57 + 5700$

    $= 5757$

14. $(57 \times 5) + (57 \times 5) = 57 \times (5 + 5) = 57 \times 10$

Chapter 9

DIVISION

In the preceding chapter a rectangular array of $\underline{a}$ rows with $\underline{b}$ members in each row was used as a physical model for $a \times b$. From this and from other models, the properties of multiplication for whole numbers were developed. We saw that multiplication of whole numbers has the properties of closure, commutativity and associativity, and that multiplication is distributive over addition. Also, the numbers 1 and 0 have the special properties that

$$1 \times a = a \times 1 = a, \text{ and}$$
$$0 \times a = a \times 0 = 0.$$

The first three properties exactly parallel the same three properties for addition, and 1 plays a role for multiplication closely corresponding to that of 0 for addition. The similarity in behavior of the two operations leads to the question as to whether there is an operation which bears to multiplication a similar relation as subtraction does to addition; namely, an inverse or undoing operation. The answer to this is the operation called division.

The procedure adopted for the study of the properties of subtraction using small whole numbers before looking at the techniques for adding and subtracting large numbers will be followed in developing properties of division using small whole numbers. The techniques of multiplication and division for large numbers will be considered in the next two chapters.

In the operation of multiplication applied to the ordered pair (4, 5), in order to determine the unknown number $\underline{n}$ which is the product 4×5 of the two known factors 4 and 5, we counted the number of members in a 4 by 5 array--that is, in an array of 4 rows with 5 members in each row (or 4 disjoint sets with 5 members in each set).

An associated problem is to start with 20 objects and ask how many disjoint subsets there are in this set if each subset is to have 4 members. In terms of arrays, the question is "if a set of 20 members is arranged 4 to a row, how many rows will there be?" In this particular case, the answer is 5, but in many cases there would be no answer; for example, 20 objects arranged 6 to a row does not give an exact number of rows.

x x x x
x x x x
x x x x
x x x x
x x x x

Figure 9-1. 20 objects arranged 4 to a row.

It is true that ordinarily we do carry out such a division process as 20 divided by 6 obtaining a quotient and a remainder. In speaking of division as an operation in the set of whole numbers, the expression, 20 divided by 6, is meaningless because it is not a whole number. The process as indicated by $\begin{array}{r}3\\6\overline{)20}\end{array}$, remainder 2, will be more fully developed later when the techniques of division are discussed in detail. It will then be pointed out that for any ordered pair (a, b) with $b \neq 0$, the dividend a may be expressed as follows:

$a = (q \times b) + r$, where $\underline{a}$ is the dividend,
$\underline{q}$ is the quotient,
$\underline{b}$ is the divisor,
and $\underline{r}$ is the remainder.

So, for the pair (20, 6),

$$20 = (3 \times 6) + 2.$$

We may identify the quotient 3 as the largest number of complete rows in an array of 20 objects, 6 to a row. In fact, the statement, $20 = (3 \times 6) + 2$, is precisely the procedure for checking the division process indicated by $\begin{array}{r}3\\6\overline{)20}\end{array}$, remainder 2.

Closure

The operation of division applied to the ordered pair of numbers (20, 4) means that an unknown factor $\underline{n}$ must be determined such that if 4 and $\underline{n}$ are the two factors, the product will be 20. That is, it is a number $\underline{n}$ that will make either one of the two number sentences $4 \times n = 20$ or $n \times 4 = 20$ a true statement. The two number sentences, of course, say the same thing since for any whole number $\underline{n}$, $4 \times n = n \times 4$. Under the operation of division, to the ordered pair (20, 4) is attached the whole number 5. For the ordered pair (20, 6) there is no such number that can be attached; nor is there for (5, 15). So, under the operation of division, (20, 6) or (5, 15) are not defined in the set of whole numbers. Division therefore does not have the property of closure in the set of whole numbers. The last case for (5, 15) is simply an example of the fact that in the ordered pair of whole numbers (a, b), if $b > a$, and $a \neq 0$, the operation of division never yields a whole number.

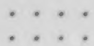
Problems *

1. What property of numbers asserts that for any whole number n, $4 \times n = n \times 4$?
2. Find the number attached to each of the following ordered pairs under the operation of division.

a.	(20, 5)	d.	(72, 9)
b.	(28, 4)	e.	(64, 8)
c.	(6, 1)	f.	(42, 7)

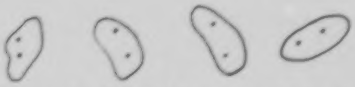
Definition of Division

The normal symbol for the operation of division is ÷ . Thus, $8 \div 2$ is the unknown factor, if there is one, which multiplied by 2 gives the product 8. It is, therefore, 4. It is also the number of columns in an array of 8 objects arranged in 2 equal rows or formed into disjoint subsets, 2 objects in each subset (see Figure 9-2).



8 objects arranged in 2 equal rows.

Figure 9-2a



Set of 8 objects in disjoint subsets, 2 objects in each subset.

Figure 9-2b

As another example, $17 \div 4$ is the number n, if any, for which $n \times 4 = 17$. A few trials will suffice to indicate that there is no such whole number. Also, the attempt to arrange an array of 17 elements in equal columns of 4 members each is doomed to failure. As a matter of fact, there is no a by b array with 17 members except the 1×17 or the 17×1 array.

* Solutions for problems in this chapter are on page 104.

There are many problems where the number relationship can be recognized as that of a known product and one known factor. The number sentence to translate this relationship will express a problem in division. Thus, if 75 tulip bulbs are to be planted in equal rows of 15 each, how many bulbs will be in each row? The number sentence to express this relationship is

$$n \times 15 = 75 \quad \text{or} \quad n = 75 \div 15.$$

Since division may be described as finding an unknown factor in a multiplication problem when the product and one factor are known, if $\underline{a}$ and $\underline{b}$ are known whole numbers, $a \div b = n$ and $a = b \times n$ are two number sentences which say the same thing. This "missing factor" concept in division parallels the "missing addend" concept in subtraction where it is noted that if $\underline{a}$ and $\underline{b}$ are known whole numbers, $a - b = n$ and $a = b + n$ are two number sentences which say the same thing. Accordingly, if $b \neq 0$, division may be defined as follows:

$$a \div b = n \quad \text{if and only if} \quad a = b \times n.$$

Why $b = 0$ is to be ruled out in division will be discussed in the next section.

In the same way as subtraction is the inverse of addition, division by $\underline{n}$ may be thought of as the inverse of multiplication by $\underline{n}$. Thus,

$$(8 \times 3) \div 3 = 8 \quad \text{and} \quad (17 \times 4) \div 4 = 17.$$

However, caution must be exercised in thinking about multiplication as the inverse of division because it is true that

$$(15 \div 3) \times 3 = 15, \quad \text{while} \quad (8 \div 3) \times 3 \quad \text{is meaningless}$$

since $8 \div 3$ is not a whole number. This is similar to the caution we must exercise in this "doing and undoing" process with subtraction; thus while

$$(15 - 3) + 3 = 15 \quad \text{is perfectly acceptable,}$$
$$(5 - 13) + 13 \quad \text{is meaningless}$$

since $(5 - 13)$ is not a whole number. Of course, the restriction will be removed later when the set of whole numbers is extended to include numbers for which $8 \div 3$ and $5 - 13$ have meaning.

Problem

3. Tell whether each of the following statements is true or whether it is meaningless for whole numbers.

a. $(3 + 9) - 9 = 3$

b. $(9 + 3) - 9 = 3$

c. $(3 - 9) + 9 = 3$

d. $(3 \times 9) \div 3 = 9$

e. $(3 \div 9) \times 9 = 3$

f. $(9 \times 3) \div 3 = 9$

g. $(9 \div 3) \times 3 = 9$

The Role of 0 and 1 in Division

The operation of division was connected to the operation of multiplication by the statement that

$$a \div b = n \text{ if and only if } a = b \times n.$$

Since 0 and 1 played special roles in multiplication, it may be appropriate to pay particular attention to the two numbers in division.

If $a = 0$ and $\underline{b}$ is any number not zero, then $0 \div b$ is that number $\underline{n}$, if there is one, such that $0 = b \times n$. From the multiplication facts, $0 = b \times n$ is certainly true if $n = 0$. So,

$$0 \div b = 0 \text{ if } b \neq 0.$$

The above is true for $a = 0$ and $b \neq 0$. Now consider $a = 0$ and $b = 0$; this is the case, $0 \div 0$. By the definition of division, $0 \div 0$ is equal to that number $\underline{n}$, if there is one, for which it is true that $0 = 0 \times n$. But by the special multiplication property of 0, $0 \times n$ is equal to 0 for any whole number n whatever. Thus

$0 \div 0$ is an ambiguous symbol.

The case of $a \div 0$, where $a \neq 0$ is still another situation. This must be equal to that number $\underline{n}$ such that $a = 0 \times n$. But this is a contradiction in terms for any number $\underline{a}$ not equal to 0 since we must have $a = 0 \times n$ and $0 \times n$ is always equal to 0. For this reason, $a \div 0$ is meaningless for $a \neq 0$; that is,

$a \div 0$ is undefined.

Notice that $0 \div b = 0$ if $b \neq 0$, but that $a \div 0$ is either ambiguous or meaningless, depending on whether or not $a = 0$. In either case, division by zero is to be avoided. Thus, 0 plays a very special role with respect to division--a role that is not understood clearly by many. In summary,

$a \div b$ is ambiguous if $a = 0$ and $b = 0$;
$a \div b$ is meaningless if $a \neq 0$ and $b = 0$;
$a \div b$ is zero if $a = 0$ and $b \neq 0$.

Problem

4. Tell whether each of the following is ambiguous, meaningless, zero, or cannot be determined.

   a. $6 \div 0$

   b. $0 \div 6$

   c. $a \div 0$

   d. $0 \div a$

   e. $0 \div 0$

Using the definition it can be seen that for any $(\neq 0)$ whole number b, $1 \div b$ is not a whole number at all unless $b = 1$, while $a \div 1 = a$ for any whole number a. Consequently,

$a \div 1 = a$ for any whole number a;
$1 \div b$ is not a whole number unless $b = 1$.

In the sense that $a \div 1 = a$, the number 1 acts somewhat like an identity element for division. Unlike the identity element for multiplication in which, for any a, $1 \times a = a \times 1$, the number 1 is limited to acting as an identity element for division if it is to the right of the symbol $\div$.

Problem

5. Tell whether each of the following is a whole number, is not a whole number, or cannot be determined; if possible, name the whole number.

   a. $8 \div 4$

   b. $2 \div 4$

   c. $3 \div 3$

   d. $6 \div 0$

   e. $0 \div 132$

   f. $1 \div b$, b is a whole number and $b = 1$.

   g. $1 \div b$, b is a whole number and $b \neq 1$.

   h. $a \div b$, a and b are whole numbers and $b > a$.

   i. $0 \div b$, b is a whole number and $b \neq 0$.

   j. $a \div b$, a and b are whole numbers and $a > b$.

   k. $a \div b$, a and b are whole numbers and $a = b$.

Properties of Division

Many examples may be given to show that the whole numbers are not closed under division. For example, while $6 \div 3 = 2$, $3 \div 6$ is not a whole number. These same two examples show that $6 \div 3 \neq 3 \div 6$, hence the operation is not commutative. To see that division is not associative, again many examples may be produced. We need only one example, and such an example is the following:

$$(12 \div 6) \div 2 = 2 \div 2 = 1, \text{ but}$$
$$12 \div (6 \div 2) = 12 \div 3 = 4.$$

The different results obtained for $(12 \div 6) \div 2$ on the one hand, and for $12 \div (6 \div 2)$ on the other, shows that, in general, it is not true that $(a \div b) \div c = a \div (b \div c)$.

So far, division with respect to whole numbers has revealed itself as an operation that does not have the properties of closure, commutativity and associativity. Furthermore, division by 0 is to be avoided. To free ourselves from the impression that not much can be said about this operation, we need to consider only the important notion that division by b is the inverse of the operation of multiplication by b. That is, $(a \times b) \div b = a$, provided, of course, $b \neq 0$.

Problems

6. For which of the following is it true that $(a \div b) \div c = a \div (b \div c)$?

   a. $4 \div 2 \div 2$

   b. $4 \div 2 \div 1$

   c. $24 \div 6 \div 2$

   d. $0 \div 5 \div 1$

   e. $9 \div 9 \div 1$

   f. $9 \div 3 \div 1$

   g. $0 \div 9 \div 3$

7. From the results of the preceding exercises, under what conditions will $(a \div b) \div c = a \div (b \div c)$?

The Distributive Property

Recall that, relating multiplication to addition and subtraction was the distributive property. In a limited way, division also has a distributive property, but care is needed in using it. This is so because division is not commutative and hence it is not expected that $a \div (b + c)$ would be the same as $(b + c) \div a$. In general,

> if $b + c > a$, $a \div (b + c)$ is not a whole number unless $a = 0$, but $(b + c) \div a$ may be a whole number.

If $(b + c) \div a$ is a whole number and if $b \div a$ and $c \div a$ are whole numbers, then it is true that

$$(b + c) \div a = (b \div a) + (c \div a).$$

This is what we mean when we say that division has a limited distributive property; that it has only the right hand distributive property, and only when $(b \div a)$ and $(c \div a)$ are defined. For example,

$$\begin{aligned}(15 + 24) \div 3 &= 39 \div 3 = 13, \text{ and}\\ (15 \div 3) + (24 \div 3) &= 5 + 8 = 13,\end{aligned}$$

and thus we see that the two results are the same; that is,

$$(15 + 24) \div 3 = (15 \div 3) + (24 \div 3).$$

On the other hand,

$20 \div (2 + 5) = 20 \div 7$ is not a whole number, whereas
$(20 \div 2) + (20 \div 5) = 10 + 4 = 14$ is a whole number.
So $20 \div (2 + 5) \neq (20 \div 2) + (20 \div 5)$.

In general then, $a \div (b + c) \neq (a \div b) + (a \div c)$, but it is true that $(b + c) \div a = (b \div a) + (c \div a)$, provided $b \div a$ and $c \div a$ have meaning. Many examples may also be produced to confirm that division has the right hand distributive property over subtraction, provided each of the indicated subtraction and divisions has meaning; that is,

$(b - c) \div a = (b \div a) - (c \div a)$, provided $b - c$,
$b \div a$, and $c \div a$ are whole numbers.

As an example, we can use $(24 - 15) \div 3$ and $(24 \div 3) - (15 \div 3)$ to illustrate this point, but it must be borne in mind that this is merely an example, and does not prove our larger claim that the right hand distributive property holds whenever each indicated subtraction and division has meaning for whole numbers.

Note that 39 can also be written as $30 + 9$ and so

$$39 \div 3 = (30 + 9) \div 3 = (30 \div 3) + (9 \div 3).$$

However, if we write 39 as $25 + 14$ it would be incorrect to say that

$$(25 + 14) \div 3 = (25 \div 3) + (14 \div 3)$$

since the latter two expressions are not whole numbers. Later when we study "fractions," we shall see that, as a statement about "fractions," this will be correct. This amounts to the same type of relaxing of restrictions as when the set of whole numbers is extended so that the set of new numbers is closed under subtraction; when "fractions" are considered, the set of whole numbers is extended so that the set of new numbers is closed under division.

Incomplete Arrays

We have found that to determine $a \div b$, we may sometimes enlist the aid of a physical model in the form of an array. For example, to determine $35 \div 5$, we put the 35 elements in 5 rows and we find that this can be done with exactly seven elements in each row.

On the other hand, if an attempt were made to determine $37 \div 5$, there is no whole number n such that $37 = 5 \times n$; hence $37 \div 5$ is not defined in the set of whole numbers. However, an approach may be made guided by the procedure in setting up an array as before. Filling out 5 rows with the 37 elements, it can be seen that one element in each row requires 5 elements; 2 elements in each row requires 10 elements; 3 in each row, 15 elements, etc., until 7 elements have been displayed in each of 5 rows. The array now employs 35 of the 37 elements, with 2 elements left over. This situation can be expressed by the number sentence,

$$37 = (5 \times 7) + 2.$$

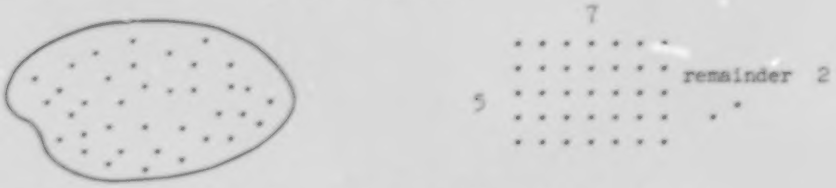


Figure 9-3. Rearrangement of 37 elements in an array.

Essentially what has been done is to set up as large an array as possible which has 5 equal rows and observe the number of elements left over. If no element remains undisplayed, then the quotient obtained is precisely the missing factor n in $b \times n = a$; otherwise, the process is said to yield a remainder.

From the physical model that had helped to determine the missing factor, the procedure has shifted to a process of obtaining a quotient and remainder, using this model as a guide. This process is then applied to such questions as 37 ÷ 5 for which it may be known that there is no missing factor among the whole numbers. This is done by expressing 37 as (7 × 5) + 2, where 7 is the quotient. In general, this is done by expressing a as (q × b) + r, where q is the quotient and r is the remainder. This can also be achieved by grouping the 37 elements in disjoint subsets of 5 elements in each subset. By this, we find 7 disjoint subsets and again with a remainder of 2 (see Figure 9-4). So,

$$37 = (7 \times 5) + 2.$$

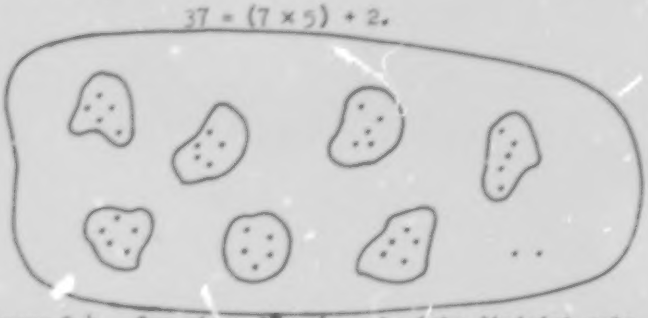


Figure 9-4. Grouping 37 elements into disjoint sets.

Note that in one regrouping we have 37 = (5 × 7) + 2 and in another regrouping, 37 = (7 × 5) + 2. Since 5 × 7 = 7 × 5, the two regroupings are equivalent; that is,

$$37 = (5 \times 7) + 2 = (7 \times 5) + 2.$$

In terms of the ordered pair (a, b), the regrouping is a = (q × b) + r, or, in the familiar form, $\begin{array}{r} q, \\ b\overline{)\,a} \end{array}$ remainder r.

Problems

8. For (37, 5) name the q and the r in the sentence 37 = (7 × 5) + 2.
9. For each of the following ordered pairs, express the pair (a, b) as a = (q × b) + r.
   a. (20, 3) b. (20, 4) c. (3, 7)
10. For each of the above number pairs, name the quotient q and the remainder r.
11. By what property of whole numbers is it true that

$$(5 \times 7) + 2 = (7 \times 5) + 2?$$

12. What is the basis for checking by the procedure,

$$\begin{array}{r} 7 \\ \times\, 6 \\ \hline 42 \\ +\, 3 \\ \hline 45 \end{array} \qquad \text{for the problem} \quad \begin{array}{r} 6 \\ 7\overline{)45} \end{array} \ \text{r}\ 3 \quad ?$$

13. Tell whether each of the following is more readily visualized by a rectangular array of 7 rows or by disjoint subsets with 7 in each subset.

    a. 42 pieces of candy are to be divided equally among 7 children.

    b. 42 pieces of candy are to be packaged 7 pieces to a package.

14. Explain why 89 tea cups cannot be packaged in 7 equal sets.

15. A standard deck of 52 playing cards is to be dealt to 3 players. Write a number sentence telling how many cards each player is to receive if the entire deck is to be dealt out as much as possible.

16. A standard deck of 52 cards is to be dealt to 7 players. Each player is to be dealt 5 cards. Write a number sentence telling how many cards are to be dealt and how many cards are to be left in the deck. How many more cards may each player be dealt?

Division of whole numbers is defined as an operation in which two given whole numbers are combined to produce a third whole number. We have found that it may or may not be possible to produce a third whole number depending on the numbers we start with. Even if division is not possible as such an operation, there is still a process that can be used in obtaining a quotient and a remainder. Certain properties of division as an operation were investigated in this chapter and will be useful in developing certain computational techniques. For example, in Chapter 11, when the techniques of long division are studied, we shall see that the distributive property will be of great value.

Exercises - Chapter 9

1. Rewrite each mathematical sentence below as a division sentence. Find the unknown factor.

    a. $5 \times n = 20$
    b. $p \times 4 = 28$
    c. $n \times 1 = 6$
    d. $9 \times n = 72$
    e. $8 \times n = 64$
    f. $q \times 7 = 42$

2. Which of the symbols below are meaningless? Which are ambiguous? Which are zero?

    a. $8 \div 0$
    b. $0 \div 4$
    c. $0 \div 0$
    d. $(4 \times 0) \div 6$
    e. $4 \times (0 \div 6)$
    f. $(4 \div 0) \times 6$
    g. $(5 \times 0) \div (7 \times 0)$
    h. $9 \div 0$
    i. $0 \div (4 + 9)$
    j. $(3 \times 0) \div 3$

3. A marching band always forms an array when it marches. The leader likes to use many different formations. Aside from the leader, the band has 59 members. The leader is trying very hard to find one more member. Why?

4. Does division have the commutative property? Give an example to substantiate your answer.

5. Give five illustrations of the distributive property for division.

6. Which of the following is distributive for whole numbers?
    - a. $(15 + 5) \div 5$
    - b. $(5 + 15) \div 5$
    - c. $5 \div (5 + 15)$
    - d. $(14 + 6) \div 4$
    - e. $4 \div (14 + 6)$
    - f. $(15 - 5) \div 5$
    - g. $(5 - 15) \div 15$
    - h. $(15 - 15) \div 15$

7. Explain whether $a \div b$ is ever a whole number, if $b > a$.

Solutions for Problems

1. Commutative property of multiplication
2. a. 4
    - b. 7
    - c. 6
    - d. 8
    - e. 8
    - f. 6
3. a. True
    - b. True
    - c. Meaningless
    - d. True
    - e. Meaningless
    - f. True
    - g. True
4. a. Meaningless
    - b. Zero
    - c. Cannot be determined; ambiguous if $a = 0$ and meaningless if $a \neq 0$.
    - d. Cannot be determined; ambiguous if $a = 0$ and zero if $a \neq 0$.
    - e. Ambiguous.

5. a. Whole number; 2
   b. Not a whole number
   c. Whole number; 1
   d. Not a whole number
   e. Whole number; 0
   f. Whole number; 1
   g. Not a whole number
   h. Cannot be determined; zero if $a = 0$ and not a whole number if $a \neq 0$.
   i. Whole number; 0
   j. Cannot be determined; meaningless if $b = 0$, the whole number a if $b = 1$, and not a whole number if $b > 1$ unless a is a multiple of b.
   k. Cannot be determined; ambiguous if $a = b = 0$ and the whole number 1 if $a = b \neq 0$.

6. a. False
   b. True
   c. False
   d. True
   e. True
   f. True
   g. True

7. If $a = 0$, or $c = 1$, or both $a = 0$ and $c = 1$.

8. $q = 7$; $r = 2$

9. a. $20 = (6 \times 3) + 2$
   b. $20 = (5 \times 4) + 0$
   c. $3 = (0 \times 7) + 3$

10. a. $q = 6$; $r = 2$
    b. $q = 5$; $r = 0$
    c. $q = 0$; $r = 3$

11. Commutative property of multiplication

12. For any ordered pair of whole numbers (a, b) when $b \neq 0$, a q and an r may be found such that $a = (q \times b) + r$.

13. a. 7 rows, 6 members to a row.
    b. 6 disjoint subsets, 7 in each subset.

14. $89 \div 7$ is not a whole number.

15. $52 = (17 \times 3) + 1$

16. $52 = (5 \times 7) + 17$; 2 more cards may be dealt to each player.

Chapter 10

TECHNIQUES OF MULTIPLICATION AND DIVISION

Computing with Large Numbers

The properties of the operations which have been so carefully set forth will now be used to develop techniques for computing the results of the operations of multiplication and division of whole numbers when the numbers are so large that the results are not immediately available from memory. For example, knowing that $a \times b$ is the number of elements in an $\underline{a}$ by $\underline{b}$ array is of little help if we are asked to compute 275×352. It is just too much trouble to count so many elements. But we can make use of the commutative and associative properties of multiplication and addition, the distributive property, the special properties of 0 and 1 as factors, the multiplication and addition facts for small numbers which we assume known and our decimal (base ten) system of numeration (see Chapter 5) to help us in our computation. If you have ever tried to multiply two numbers, even fairly small ones, which are written as numerals in the Roman system you will appreciate more than ever our own place value system. Try (XVII) × (DCXI). The properties of numbers do not depend on the numeration system we use for naming them, but facility in computation does lean heavily on it.

The process of computing a number such as 7×24 depends on the idea of renaming 24 as the sum of two smaller numbers so that we can compute by the ordinary table of known "multiplication facts." When 24 has been so renamed, we apply the distributive property; thus we might say

$$7 \times 24 = 7 \times (9 + 15) = (7 \times 9) + (7 \times 15)$$

and in turn this is equal to

$$\begin{aligned}(7 \times 9) + [7 \times (8 + 7)] &= (7 \times 9) + [(7 \times 8) + (7 \times 7)] \\ &= 63 + 56 + 49 = 168.\end{aligned}$$

There are other ways of renaming 24 and if we try several of them we soon discover that the way which makes our computation easiest is to write 24 in the standard expanded form with base 10. That is, we write $24 = (2 \times 10) + 4$. We may write out the details of the computation naming at each step the property which justifies it. Then $7 \times 24 =$

$$\begin{aligned}
7 \times [(2 \times 10) + 4] &= [7 \times (2 \times 10)] + (7 \times 4), && \text{distributive} \\
&= [(7 \times 2) \times 10] + 28, && \text{associative} \\
&= (14 \times 10) + 28, && \text{renaming } 7 \times 2 \text{ as } 14 \\
&= [(10 + 4) \times 10] + (20 + 8), && \text{expanding base } 10 \\
&= (10 \times 10) + (4 \times 10) + (2 \times 10) + 8, && \text{distributive} \\
&= (10 \times 10) + [(4 + 2) \times 10] + 8, && \text{distributive} \\
&= (1 \times 100) + (6 \times 10) + 8, && \text{renaming} \\
&= 168, && \text{renaming.}
\end{aligned}$$

Of course, we never go through this amount of detail when actually doing problems. Normally this is condensed as follows:

$$\begin{aligned}
7 \times 24 &= 7 \times (20 + 4) \\
&= (7 \times 20) + (7 \times 4) \\
&= 140 + 28 \\
&= 168
\end{aligned}$$

Problems*

1. Rewrite 24 as the sum of three numbers, each less than 9 and show the computation involved in 7×24.
2. Use the condensed form as shown in the example, $7 \times 24 = 7 \times (20 + 4)$, to show the computation for 7×42.

Notice that the computation shown by the condensed form is actually being done when the multiplication is put in vertical form but that certain abbreviations and omissions are made to save writing. Thus,

$$\begin{array}{r} 24 \\ \underline{\times\ 7} \end{array} \quad \text{may be written as} \quad \begin{array}{r} 20 \\ \underline{\times\ 7} \\ 140 \end{array} \; + \; \begin{array}{r} 4 \\ \underline{\times\ 7} \\ 28 \end{array} \; = \; 168$$

or more usually as

$$\begin{array}{rl} 24 & \\ \underline{\times\ 7} & \\ 28 & (= 7 \times 4) \\ \underline{140} & (= 7 \times 20) \\ 168 & \end{array} \quad \text{which may be shortened to} \quad \begin{array}{r} 24 \\ \underline{\times\ 7} \\ 28 \\ \underline{14} \\ 168 \end{array}$$

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*Solutions for problems in this chapter are on page 116.

or even, if one can remember the "2 tens" from $28 = (2 \times 10) + 8$ and add it mentally to the "4 tens" from $140 = (1 \times 100) + (4 \times 10)$, we can write

$$\begin{array}{r} 24 \\ \times\ 7 \\ \hline 168 \end{array}$$

These same procedures may be used for finding the products of pairs of larger numbers. The thought process and the record of results become more and more complex.

$$\begin{array}{lrll} \text{Suppose} & n & = 20 \times 34; & \\ \text{then} & 20 \times 34 & = 20 \times (30 + 4), & \text{since } 34 = 30 + 4 \\ & & = [20 \times 30] + [20 \times 4], & \text{distributive} \\ & & = [(2 \times 10) \times 30] + [(2 \times 10) \times 4], & \text{since } 20 = 2 \times 10 \\ & & = [(2 \times 10) \times (3 \times 10)] + [(2 \times 10) \times 4], & \text{distributive.} \end{array}$$

By using the associative and commutative properties we know that

$$(2 \times 10) \times (3 \times 10) = (2 \times 3) \times (10 \times 10) = 6 \times 100, \quad \text{and}$$
$$(2 \times 10) \times 4 = 4 \times (2 \times 10) = (4 \times 2) \times 10 = 8 \times 10.$$

We have therefore:

$$20 \times 34 = (6 \times 100) + (8 \times 10)$$

which by our decimal system of numeration can be written as 680. Notice that the vertical form of this multiplication carries out the same ideas though in much condensed form,

$$\begin{array}{rl} 34 & \\ \times\ 20 & \\ \hline 80 & (= 20 \times 4) \\ \underline{600} & (= 20 \times 30) \\ 680 & \end{array}$$

Let's try 34×46.

$$\begin{array}{rll} 34 \times 46 & = (30 + 4) \times 46 = (30 \times 46) + (4 \times 46), & \text{distributive} \\ & = [30 \times (40 + 6)] + [4 \times (40 + 6)] & \\ & = (30 \times 40) + (30 \times 6) + (4 \times 40) + (4 \times 6), & \text{distributive} \\ & = 1200 + 180 + 160 + 24 & \\ & = 1564 & \end{array}$$

Written in a sort of expanded vertical form the computation looks like this:

$$\begin{array}{r} 46 \\ \times\ 34 \\ \hline \end{array} = \begin{array}{r} (40 + 6) \\ \times\ (30 + 4) \\ \hline \end{array} = \begin{array}{r} 40 + 6 \\ \times\ 30 \\ \hline \end{array} + \begin{array}{r} 40 + 6 \\ \times\ 4 \\ \hline \end{array}$$

$$= \begin{array}{r} 40 \\ \times\ 30 \\ \hline 1200 \end{array} + \begin{array}{r} 6 \\ \times\ 30 \\ \hline 180 \end{array} + \begin{array}{r} 40 \\ \times\ 4 \\ \hline 160 \end{array} + \begin{array}{r} 6 \\ \times\ 4 \\ \hline 24 \end{array} = 1564$$

In normal vertical form, but still unabbreviated, it looks like this:

$$\begin{array}{rl} 46 & \\ 34 & \\ \cline{1-1} 24 & (4 \times 6) \\ 160 & (4 \times 40) \\ 180 & (30 \times 6) \\ 1200 & (30 \times 40) \\ \cline{1-1} 1564 & \end{array}$$

This is usually shortened to read:

$$\begin{array}{r} 46 \\ 34 \\ \hline 184 \\ 1380 \\ \hline 1564 \end{array} \quad \text{and sometimes to} \quad \begin{array}{rl} 46 & \\ 34 & \\ \hline 184 & \\ 138 & \\ \hline 1564 & \end{array}$$

Estimates of the Product

Sometimes, particularly if you are interested in estimates rather than exact answers, you can think like this:

$$\begin{aligned} 38 \times 43 &= (30 + 8) \times 43 = (30 \times 43) + (8 \times 43) \\ &= (30 \times 40) + (30 \times 3) + (8 \times 40) + (8 \times 3) \\ &= (40 \times 30) + (3 \times 30) + (40 \times 8) + (3 \times 8). \end{aligned}$$

A rough estimate of the answer would be 30×40; for a better estimate, add 40×8 to the rough estimate; and for a still better estimate, add 30×3 to the second estimate.

$$\begin{array}{lrrrrr} & 38 & & & & \\ & \times\ 43 & & & & \\ \cline{2-2} (40 \times 30) & 1200 & 1200 & 1200 & 1200 & 1200 \\ (40 \times \ 8) & 320 & & 320 & 320 & 320 \\ (\ 3 \times 30) & 90 & & & 90 & 90 \\ (\ 3 \times \ 8) & 24 & & & & 24 \\ \cline{2-6} & 1634 & 1200 & 1520 & 1610 & 1634 \\ & & \text{rough} & \text{better} & \text{still} & \text{exact} \\ & & \text{estimate} & \text{estimate} & \text{better} & \text{answer} \end{array}$$

Notice that in this case we start multiplying with the digits on the left which represent the larger parts of the given factors.

Notice also the positional system coming into play in each of the intermediate computations. For example,

$$\begin{aligned} 40 \times 30 &= 4 \times 10 \times 3 \times 10 \\ &= 4 \times 3 \times 10 \times 10 \\ &= 12 \times 100 \\ &= (10 + 2) \times 100 \\ &= (10 \times 100) + (2 \times 100) \\ &= (1 \times 1000) + (2 \times 100) \end{aligned}$$

This indicates a 12 in the hundreds' position, or a 1 in the thousands' position and a 2 in the hundreds' position. The pattern of the number of zeros should also be noted in the multiplication process.

Problems

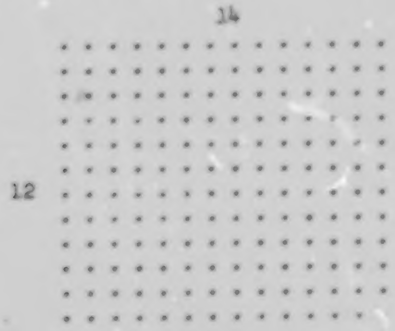
3. Show a rough estimate and show successively better estimates for
   a. 43 × 21
   b. 76 × 58
4. How many zeros follow the 8 in 200 × 40,000?
5. Write the number sentence which states that one-hundred thousands is one one-hundred-thousand.
6. Since 2 × 5 = 10, how many zeros follow the 1 in 20,000 × 50,000? In which position is the 1 in 20,000 × 50,000?

To multiply three digit or larger numbers, the same techniques apply. For example, if

$$\begin{aligned} n &= 234 \times 433, \quad \text{we may write} \\ n &= (200 + 30 + 4) \times 433 \\ &= (200 \times 433) + (30 \times 433) + (4 \times 433) \\ &= (4 \times 433) + (30 \times 433) + (200 \times 433) \end{aligned}$$

$$\begin{array}{r} 433 \\ \times\ 234 \\ \hline 1732 \\ 12990 \\ 86600 \\ \hline 101322 \end{array} = 234 \times 433$$

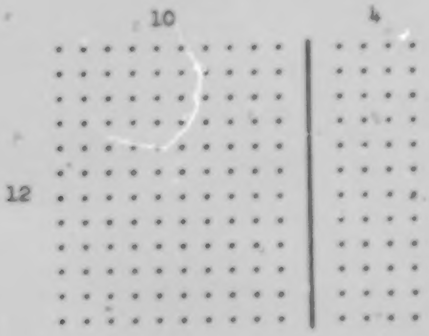
It might be well at this time to go back to our definition of the production of 12 × 14 as the number of elements in an array of 12 rows of 14 elements per [illegible] and see how our computation process is reflected in our array. First we represent the array:



12 × 14

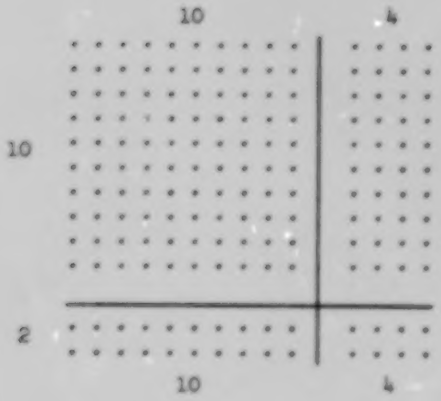
Figure 10-1.

We separate it into several smaller arrays:



$$12 \times (10 + 4) = (12 \times 10) + (12 \times 4)$$

Figure 10-2.



$$12 \times 14 = 12 \times (10 + 4) = (12 \times 10) + (12 \times 4)$$
$$= (10 + 2) \times 10 + (10 + 2) \times 4$$
$$= (10 \times 10) + (2 \times 10) + (10 \times 4) + (2 \times 4)$$

Figure 10-3.

The four arrays in Figure 10-3 portray the four partial sums which the distribution of the multiplications gives.

Also the results now can be put in the form

$$(1 \times 100) + [(2 \times 10) + (4 \times 10)] + (2 \times 4) = (1 \times 100) + (6 \times 10) + 8,$$

and the product can be easily read as 168 in our decimal numeration.

Division Algorithm

We have noted in Chapter 9 that the whole numbers are not closed under the operation of division. Thus, for example, for the ordered pair (37, 7), division as an operation does not yield a whole number. The attempt to set up an array of 37 elements in 7 rows fails, but in the attempt, we are led to a division process which gives us a quotient and a remainder--namely 5 and 2. This process is called the "division algorithm" (after the ninth century Arabian mathematician Al-Kworesmi who wrote the first book on arithmetic algorithms). "Algorithm," or "algorism," as it is sometimes spelled, is a technical word in mathematics which means a numerical process that may be applied again and again to reach a solution of a problem. For $a \div b$, the division algorithm then eventually gives us the largest whole number $\underline{q}$ such that $q \times b \leq a$. If $q \times b = a$, then $\underline{q}$ is the missing factor. If $q \times b < a$, then $\underline{q}$ is the quotient and there is a remainder. So $a = (q \times b) + r$, where $\underline{r}$ may or may not be 0.

Let us examine more closely what occurs as we attempt to set up an array of 37 elements in 7 rows. First, from the set of 37 elements, we obtain 7 elements and place these 7, one to a row. In the original set of the 37 elements now remain 30 elements. We proceed to obtain another 7 elements from the remaining elements, etc. This process is indicated as follows:

$$\begin{array}{rl}
37 & \text{elements in the original set} \\
\underline{-\ 7} & \\
30 & \text{elements remain after displaying } 1 \text{ to a row} \\
\underline{-\ 7} & \\
23 & \text{elements remain after displaying } 2 \text{ to a row} \\
\underline{-\ 7} & \\
16 & \text{elements remain after displaying } 3 \text{ to a row} \\
\underline{-\ 7} & \\
9 & \text{elements remain after displaying } 4 \text{ to a row} \\
\underline{-\ 7} & \\
2 & \text{elements remain after displaying } 5 \text{ to a row}
\end{array}$$

This process is referred to as the repeated subtraction description of division and parallels the repeated addition description of multiplication. The final step shows that the quotient is 5 and the remainder is 2. Notice that each step may be described as follows:

$$\begin{array}{lr}
 & 37 \\
 & \underline{-\ 7} \\
37 = (1 \times 7) + 30 & 30 \\
 & \underline{-\ 7} \\
37 = (2 \times 7) + 23 & 23 \\
 & \underline{-\ 7} \\
37 = (3 \times 7) + 16 & 16 \\
 & \underline{-\ 7} \\
37 = (4 \times 7) + 9 & 9 \\
 & \underline{-\ 7} \\
37 = (5 \times 7) + 2 & 2
\end{array}$$

or as $\begin{array}{r}1\\7\overline{)37}\end{array}$ r 30, $\begin{array}{r}2\\7\overline{)37}\end{array}$ r 23, ... , $\begin{array}{r}5\\7\overline{)37}\end{array}$ r 2. In the example, $\begin{array}{r}4\\7\overline{)37}\end{array}$ r 9, the 4 is sometimes referred to as the partial quotient; as is the 3 in $\begin{array}{r}3\\7\overline{)37}\end{array}$ r 16, etc. In the algorithm, what we are looking for is the largest number of complete columns possible in the array; in this case it is 5, so 5 is the quotient. In the chain of repeated subtractions we get closer and closer to the quotient desired. We can imagine, then, the same process applied to $a \div b$; ultimately, we want to get the largest number $\underline{q}$ such that $a = (q \times b) + r$.

It is not likely that we would want to go through all this manipulation every time we have to compute $a \div b$, but for the youngsters in the lower grades, this kind of procedure puts computation in a manageable size and can be refined as the youngsters acquire more and more experience with the algorithm.

Exercises - Chapter 10

1. Use the commutative, associative and distributive properties of multiplication to do the examples below. Follow the form of the illustration.

$20 \times 37 = 20 \times (30 + 7)$	[Think of 37 as 30 + 7]
$= (20 \times 30) + (20 \times 7)$	[Distributive Property]
$= 600 + 140$	[Write the products]
$= 740$	[Addition]

a. 40×30
b. 42×30
c. 76×80
d. 90×57
e. 50×76
f. 52×47

2. In the example to the right, explain why the 866 on the fifth line does not represent 2×433.

$$\begin{array}{r} 433 \\ \times\ 234 \\ \hline 1732 \\ 1299 \\ 866 \\ \hline 101322 \end{array}$$

3. For the above example, write out the full decimal expansion of 4×433 to get 1732.

4. By the distributive property, show how 4×433 may be considered the repeated addition of four 433's.

5. Use the repeated subtraction process to describe each of the following; then write each dividend in the form: $a = (q \times b) + r$.

a. $47 \div 8$
b. $28 \div 7$

6. How many times must 4 be subtracted from 95 in the algorithm indicated by $95 \div 4$?

Solutions for Problems

1. $$\begin{aligned} 7 \times 24 = 7 \times (8 + 8 + 8) &= 7 \times [8 + (8 + 8)] \\ &= (7 \times 8) + [7 \times (8 + 8)] \\ &= (7 \times 8) + (7 \times 8) + (7 \times 8) \\ &= 56 + 56 + 56 \\ &= 168 \end{aligned}$$

2. $$\begin{aligned} 7 \times 42 = 7 \times (40 + 2) &= (7 \times 40) + (7 \times 2) \\ &= 280 + 14 \\ &= 294 \end{aligned}$$

3. a. $$\begin{array}{rrl} 20 \times 40 = & 800 & \text{rough estimate} \\ 20 \times \ \ 3 = & \underline{+60} & \\ & 860 & \text{better estimate} \\ 1 \times 40 = & \underline{+40} & \\ & 900 & \text{still better} \\ 1 \times \ \ 3 = & \underline{+\ 3} & \\ & 903 & \text{exact answer} \end{array}$$

   b. $$\begin{array}{rrl} 50 \times 70 = & 3500 & \text{rough estimate} \\ 50 \times \ \ 6 = & \underline{+300} & \\ & 3800 & \text{better estimate} \\ 8 \times 70 = & \underline{+560} & \\ & 4360 & \text{still better} \\ 8 \times \ \ 6 = & \underline{+\ 48} & \\ & 4408 & \text{exact answer} \end{array}$$

4. 6 zeros

5. $100 \times 1000 = 1 \times 100{,}000$

6. 9 zeros; the 1 is in the billions' position.

Chapter 11

DIVISION TECHNIQUES

Uniqueness of q and r

In the last chapter we observed that division as an operation for whole numbers may or may not yield a whole number. Whether or not $a \div b$ was defined in the set of whole numbers, we nonetheless did arrive at a *process* which is called the division *algorithm*. The algorithm gave us the largest whole number q such that $q \times b \leq a$, and enabled us to express the dividend as

$$a = (q \times b) + r, \quad \text{provided} \quad b \neq 0 \quad \text{and} \quad r < b.$$

Throughout the discussion, when we spoke of *the* quotient and *the* remainder, we had been assuming implicitly that there was only one such q and only one such r, and our experience with division of whole numbers had never led us to believe otherwise. As a matter of fact, it can be shown that for any pair (a, b) with $b \neq 0$, the quotient q and the remainder r are uniquely determined. By this we mean there is one and only one such q and one and only one such r, where $r < b$. Before showing this, let's look at an example such as $(37, 5)$. The number 37 is represented on the number line by one and only one point; so is the number 5 $(= 1 \times 5)$; so are the numbers 10 $(= 2 \times 5)$, 15 $(= 3 \times 5)$, etc.; so is 0 $(= 0 \times 5)$. See Figure 11-1.

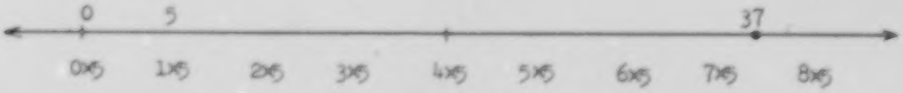


Figure 11-1. Determining q and r for $37 \div 5$.

The numbers $0, 5, 10, \ldots,$ form an ordered set of multiples of 5:

$$\{0, 5, 10, 15, \ldots\}.$$

If we start to represent some of these on the number line, we shall see that 0 is to the left of 37; 5 is to the left of 37; 10 is to the left of 37; 15 is to the left of 37; etc. Eventually we get to a point that represents the first one of the multiples of 5 represented on the number line to the right of 37. In this case, it is 8×5 or 40. The multiple of 5 just preceding this is the one we are interested in; namely, 7×5. Remember that considering division in terms of an array, this is the largest 5 row array that can be completed for 37, and that the number of elements, 7, is the quotient. 35 (or 7×5) is the largest multiple of 5 not exceeding 37. Returning to the number line, 35 is represented by one and only one point. From this, the quotient q $(= 7)$ is uniquely determined and the remainder r $(= 37 - 35 = 2)$ is uniquely determined.

In general, for any pair (a, b) such that $b \neq 0$, the same procedure can be followed. The number a is represented on the number line by one and only one point; so is $1 \times b$; so are the numbers $2 \times b$, $3 \times b$, etc.; so is $0 \times b$. Eventually, there will be a first one of the numbers in the ordered set

$$\{0 \times b, 1 \times b, 2 \times b, \ldots\},$$

that is to the right of a in the number line. Let's call the multiple of b just preceding this $q \times b$. There is one and only one point representing $q \times b$, hence q is uniquely determined.

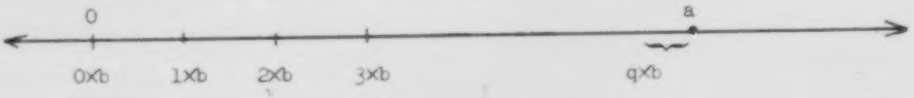


Figure 11-2. Determining q and r on the number line.

If $q \times b$ coincides with a, then q is the missing factor such that $a = q \times b$; here, the remainder is 0. That is, $a = (q \times b) + 0$. If $q \times b$ does not coincide with a, then there will be a remainder. Since $q \times b$ and a are whole numbers, the difference, $a - (q \times b)$, is a whole number. This is the remainder r, and $a = (q \times b) + r$. The numbers a and $q \times b$ are uniquely determined, hence, so is r.

Computing with Large Numbers

For small numbers we can usually see whether or not for $a \div b = n$ or for the equivalent sentence $b \times n = a$, there is an unknown factor n which multiplied by the known factor b will give the known product a. Thus, $18 \div 6 = n$ has the solution 3 since $6 \times 3 = 18$, and $67 \div 13$ has no solution in whole numbers. It is not obvious, however, whether $648 \div 24$ has a whole number solution or not.

For the expression, $67 \div 13$, we find that we can put 13 elements, one in each of the 13 rows of our array and have 54 left. We can put out another 13 and have 41 left. Again, we put out 13 and have 28 left. Next time we have 15 left and then 2. We have put 5 elements in each row and have 2 left. So we can write $67 = (5 \times 13) + 2$. Notice that in doing the problem we repeatedly subtracted 13 elements. Instead of doing this repeated subtraction of 13 elements we might try to see how large a multiple of 13 we could subtract from 67 at once.

If we guess 6 we find that $6 \times 13 = 78$ and we cannot subtract 78 from 67. If we try 4 we can't go wrong, but we do not get our result immediately.

$67 = 52 + 15 = (4 \times 13) + 15$ but $15 > 13$ so we can still write

$$\begin{aligned} 67 &= (4 \times 13) + (1 \times 13) + 2 \\ 67 &= [(4 \times 13) + (1 \times 13)] + 2 \\ 67 &= (5 \times 13) + 2. \end{aligned}$$

Our best choice would be 5 since $5 \times 13 = 65$ and we can write directly

$$67 = 65 + 2 = (5 \times 13) + 2.$$

Notice what we are attempting to do. If we cannot get at the quotient immediately, we split the dividend 67 into two addends so that the divisor 13 is a factor of at least one of the addends. Thus, 67 may be expressed as $52 + 15$, where 13 is a factor of 52. We could have expressed 67 as $26 + 41$, where 13 is a factor of 26. The point is that we simply want 13 to be a factor of at least one of the addends.

In the example where 67 is split into $52 + 15$, the divisor 13 is a factor of one of the addends, namely 52. The other addend, 15, is greater than the divisor. We can express 15 as $13 + 2$, where again the divisor 13 is a factor of the first addend 13. This process can continue until any addend either has the divisor as a factor, or is less than the divisor. In our example,

$$67 = 52 + 13 + 2.$$

Both 52 and 13 have 13 as a factor; 2 is less than 13. The number sentence may now be restated

$$\begin{aligned} 67 &= (4 \times 13) + (1 \times 13) + 2 \\ &= [(4 \times 13) + (1 \times 13)] + 2 \\ &= (5 \times 13) + 2. \end{aligned}$$

When the dividend is split into two addends so that the divisor is a factor of at least one of these addends, it may turn out that the divisor is also a factor of the other addend. For example, in $65 \div 13$, we may write

$$65 = 39 + 26.$$

In this case, 13 is a factor of both 39 and 26; then we may apply the right hand distributive property to get the missing factor. Thus,

$$\begin{aligned} 65 &= 39 + 26 = (3 \times 13) + (2 \times 13) \\ &= (3 + 2) \times 13 = 5 \times 13, \end{aligned}$$

and 5 is the missing factor; so $65 \div 13 = 5$.

To illustrate further, if we want to try to find a number $\underline{n}$ such that $165 \div 15 = n$, we can write

$$\begin{aligned} 165 &= 150 + 15 \\ &= (10 \times 15) + (1 \times 15). \end{aligned}$$

Then

$$\begin{aligned} 165 \div 15 &= [(10 \times 15) + (1 \times 15)] \div 15 \\ &= [(10 \times 15) \div 15] + [(1 \times 15) \div 15]. \end{aligned}$$

By the definition of division as the inverse of multiplication,

$$(10 \times 15) \div 15 = 10 \text{ and } (1 \times 15) \div 15 = 1.$$

$$\text{So, } 165 \div 15 = 10 + 1 = 11.$$

If we try the same process with $191 \div 15$, we may notice that

$$191 = 150 + 41.$$

150 has 15 as a factor; the other addend, 41, is greater than 15. So we continue to split 41 into addends, for example, into $30 + 11$. Now $191 = 150 + 41 = (150 + 30) + 11$, and while we know that both 150 and 30 have 15 as a factor since

$$150 \div 15 = 10 \text{ and } 30 \div 15 = 2,$$

we also know that $11 \div 15$ is not a whole number. It is evident that we cannot apply the distributive property of the division operation to the whole sum. Nevertheless, the part of the sum in parentheses, $(150 + 30)$, has 15 as a factor and we can say

$$\begin{aligned} (150 + 30) \div 15 &= (150 \div 15) + (30 \div 15) \\ &= 10 + 2 \\ &= 12. \end{aligned}$$

Now using the facts that $(a \div b) \times b = a$ (if $a \div b$ is a whole number) and $(150 + 30) \div 15 = 12$, we can write

$$\begin{aligned}
191 &= (150 + 30) + 11 \\
191 &= [(150 + 30) \div 15] \times 15 + 11 \\
191 &= (12 \times 15) + 11 .
\end{aligned}$$

The computation process is to subtract from 191 as large a multiple of 15 as is easily recognized and to repeat this process as many times as possible. We write this in a vertical set-up as follows:

$$\begin{array}{lr}
 & 2 \\
 & 10 \\
 & 15\overline{)191} \\
(10 \times 15) & \underline{150} \\
 & 41 \\
(2 \times 15) & \underline{30} \\
 & 11
\end{array}$$

We were able to subtract 15, (10 + 2) times before getting a remainder which was less than 15.

$$\begin{aligned}
191 &= [15 \times (10 + 2)] + 11 \\
&= (15 \times 12) + 11
\end{aligned}$$

Suppose we want to try $575 \div 25$. We can write it in a slightly different form.

$$\begin{array}{r|ll}
25\overline{)575} & & \\
\underline{250} & 10 & (250 = 25 \times 10) \\
325 & & \\
\underline{250} & 10 & (250 = 25 \times 10) \\
75 & & \\
\underline{75} & 3 & (75 = 25 \times 3) \\
\hline
0 & 23 = 10 + 10 + 3 &
\end{array}$$

This time we did not subtract as large a multiple of 25 each time as we might have so the process took a little longer, but we got the correct result. Because of the form in which the series of trials is indicated in the computation, this is sometimes referred to as the "escalator method." We might have done this:

$$\begin{array}{r|l}
25\overline{)575} & \\
\underline{500} & 20 \\
75 & \\
\underline{75} & 3 \\
\hline
0 & 23
\end{array}$$

In this case we write

$$575 = (23 \times 25) + 0.$$

Here, the object is to refine the process so that the first partial quotient takes care of the 100's, the second takes care of the 10's, etc. If the dividend were in the thousands, then the object is to refine the process so that the first partial quotient takes care of the 1000's, and so on.

To see the mechanism involved in the technique, let us examine $5439 \div 4$ and its corresponding algorithm in vertical form:

$$\begin{array}{rr|r}
4 & \overline{)5439} & \\
 & \underline{4000} & 1000 \\
 & 1439 & \\
 & \underline{1200} & 300 \\
 & 239 & \\
 & \underline{200} & 50 \\
 & 39 & \\
 & \underline{36} & \underline{9} \\
\text{remainder} & 3 & 1359
\end{array}$$

As we can see from the various steps, the dividend, 5439, is split into the addends successively thus,

$$\begin{aligned}
5439 &= 4000 + 1439 \\
&= 4000 + 1200 + 239 \\
&= 4000 + 1200 + 200 + 36 + 3.
\end{aligned}$$

Notice that in the final splitting, the 1000's, 100's and 10's all have 4 as a factor, and we can apply the right hand distributive property to at least this portion of the sum. Hence,

$$\begin{aligned}
5439 &= [(4000 + 1200 + 200 + 36) \div 4] \times 4 + 3 \\
&= [(1000 + 300 + 50 + 9) \times 4] + 3 \\
&= (1359 \times 4) + 3.
\end{aligned}$$

Problems*

1. For $15{,}139 \div 13$ show the splitting of 15,139 into addends of 1000's, 100's, etc., such that each addend greater than 13 has 13 as a factor. Show this also in the vertical (escalator) form.
2. Do the same for $40{,}728 \div 8$.
3. Show whether $648 \div 24$ has a whole number solution.

The technique we have developed is known as the "division algorithm" or the "division process." It enables us to write any ordered pair of numbers (a, b) in the form $a = (n \times b) + r$, where $r < b$, regardless whether or not $a \div b$ is defined in the set of whole numbers. If we find that $r = 0$ we have shown that there is an n such that $a = n \times b$ and therefore $a \div b = n$ and the operation of division applied to the numbers a and b yields a whole number n.

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* Solutions for problems in this chapter are on page 125.

Let us emphasize the very important fact about whole numbers which we have just established.

> If (a, b) is any ordered pair of whole numbers with $b \neq 0$, it is always possible to find two whole numbers q and r with $r < b$ such that $a = (q \times b) + r$.

What happens if $b > a$? Suppose $b = 15$ and $a = 9$. We can write $9 = (0 \times 15) + 9$, or in general $a = (0 \times b) + r$. We see that $q = 0$ and $a = r$.

In applying the division algorithm to large numbers, an ability to make a good estimate of the product of two numbers is a great help. Suppose we want to try $978 \div 37$. What we want to do is subtract multiples of 37 from 978. Shall we subtract 100×37? This is much too large. 10×37? This is all right, but the larger the number we subtract each time, the quicker we get through. An estimate of 30×37 shows that this product is over 1000 and so somewhat too large. We next try 20×37 and find this is all right. If we set the work up in the standard form for the division algorithm, the fact that we are using 20 as the multiplier is not quite obvious, so we change the form slightly to make the situation a little clearer. Thus

$$\begin{array}{r|l} 37\overline{)978} & \\ \underline{740} & \underline{\;20\;} \\ 238 & \end{array} \qquad (740 = 20 \times 37)$$

So far we have $978 = (20 \times 37) + 238$. We follow the same procedure with 238. Then

$$\begin{array}{r|l} 37\overline{)238} & \\ \underline{222} & \underline{\;6\;} \\ 16 & \end{array} \qquad (222 = 6 \times 37)$$

and $238 = (6 \times 37) + 16$. Notice how the process repeats itself. We now have

$$\begin{aligned} 978 &= (20 \times 37) + (6 \times 37) + 16 \\ &= [(20 + 6) \times 37] + 16 \\ &= (26 \times 37) + 16, \end{aligned}$$

and we are finished because $16 < 37$.

The work can be exhibited in complete form as:

$$\begin{array}{r|r} 37\overline{)978} & \\ \underline{740} & 20 \\ 238 & \\ 222 & +\ 6 \\ \hline 16 & 26 \end{array} \qquad \text{or} \qquad \begin{array}{r} \underline{26} = (20 + 6) \\ 6 \\ 20 \\ 37\overline{)978} \\ \underline{740} \\ 238 \\ \underline{222} \\ 16 \end{array}$$

but we usually write the two stages as

$$\begin{array}{r} 2 \\ 37\overline{)978} \\ \underline{74} \\ 23 \end{array} \qquad \text{and then} \qquad \begin{array}{r} 26 \\ 37\overline{)978} \\ \underline{74} \\ 238 \\ \underline{222} \\ 16 \end{array}$$

The first step conceals from many people that what we are really doing is subtracting 20×37 from 978. Of course, once we understand what is going on, all the short cuts and time saving abbreviations and omissions possible are permissible. However, we should keep in mind that the division process applied to (a, b) is essentially subtracting from $\underline{a}$ as large multiples of $\underline{b}$ as is conveniently possible until the remainder is less than $\underline{b}$ and then counting up how many such multiples have been subtracted.

Many problems have a number relationship which when put into a number sentence reveals the need of applying the division process. For example, how many 47 passenger buses will a school have to order to take its 820 pupils on a school excursion? We don't know if they will fit evenly into a certain number of buses, but we rather doubt it, so we set up the number sentence,

$820 = (q \times 47) + r$ and we find that

$820 = (10 \times 47) + 350$ so we need more than 10 buses

$350 = (5 \times 47) + 115$ so we need more than $10 + 5 = 15$ buses

$115 = (2 \times 47) + 21$ so we need 17 buses and have 21 children left. We either order 18 buses and have some empty seats or provide a smaller bus or some private cars for the 21 children remaining after 17 buses have been filled, since we don't want to leave anybody behind. More quickly, of course, we could compute

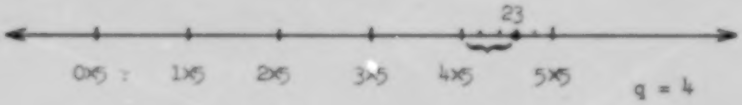
$$\begin{array}{r} 17 \\ 47\overline{)820} \\ \underline{47} \\ 350 \\ \underline{329} \\ 21 \end{array} \qquad 820 = (17 \times 47) + 21.$$

Exercises - Chapter 11

Use the division process as described in this unit for each of the ordered pair of numbers given below. Then write the mathematical sentence corresponding to $a = (q \times b) + r$ for each case.

1.	(512, 8)	6.	(14, 23)
2.	(644, 7)	7.	(23, 14)
3.	(526, 21)	8.	(720, 19)
4.	(779, 18)	9.	(50, 100)
5.	(836, 42)	10.	(6535, 47)

11. For the pair (37, 7), show the segments of 7's on the number line and locate the point representing 37. Find the quotient $\underline{q}$ and the remainder $\underline{r}$ on the number line. For example, for (23, 5),



q = 4

r = 3

12. Follow the directions in the above exercise for the pair (26, 6).

13. Do the same for the pair (3, 8).

14. Write the number sentence in the form $a = (q \times b) + r$ for the pair (0, 52).

Solutions for Problems

1. 15,139 = 13,000 + 1300 + 780 + 52 + 7

```
13/15139  |
   13000  |   1000
    2139  |
    1300  |    100
     839  |
     780  |     60
      59  |
      52  |      4
       7      1164
```

2. $40{,}728 = 40{,}000 + 720 + 8$

```
8/40728  |
  40000  |        5000
    728  |
    720  |          90
      8  |
      8  |           1
  ---------------------
      0           5091
```

3. $648 \div 24 = 27$, a whole number.

Chapter 12

SENTENCES, NUMBER LINE

In developing the properties of numbers and various operations on numbers, we have been using a rather special language involving

> symbols for numbers, such as: $1, 5, 2, 9, 3, \ldots$;
> symbols for operations, such as: $+, -, \times, \div$;
> and symbols showing relations between numbers, such as: $=, \neq, <, >, \leq, \geq$.

We have seen that a number may be named by many numerals, for example, $3 + 4, 9 - 2, \frac{28}{4}, 7$, VII, all name the same number, and we may write

$$3 + 4 = 9 - 2, \quad \text{or} \quad 9 - 2 = \frac{28}{4}, \quad \text{or} \quad 3 + 4 = 7, \quad \text{etc.},$$

to show that these are numerals for the same number. In this way, we form mathematical sentences, where the symbol, $=$, acts as the verb. The numeral to the left acts as the subject, and the numeral to the right acts as a predicate noun.

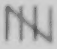
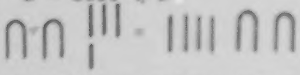
A statement as "$7 - 5 = 2$" is in mathematical form, but it can be put into words as in the sentence, "If five is subtracted from seven, the result is two." The sentence, "The result of adding the number five to the number nine is the number fourteen," can also easily be put in the much shorter form "$9 + 5 = 14$." A great deal of mathematics is in the form of sentences about numbers or number sentences as they are called. Sometimes the sentences make true statements as in both of the above examples; sometimes the number sentences are false as in the cases "$5 + 7 = 11$" or "$17 - 4 = 3$." Whether it is true or false no more disqualifies the statement as a sentence than the statement, "George Washington was vice president under Abraham Lincoln" is disqualified as a sentence.

Open Sentences

As we have noted, verbal sentences may be true: "George Washington was the first President of the United States," or false: "Abraham Lincoln was the first President of the United States." We also encounter sentences such as: "He was the first President of the United States." If read out of context, it may not be known to whom "he" referred and it may thus be impossible to determine whether the sentence is true or false. In fact,

"□ was the first President of the United States" may be a test question requiring the name of the man for which it would be a true sentence. Such a sentence is called an open sentence and is of great usefulness not only in history tests but in many other situations as well. In fact, open number sentences are the basis of a great deal of work in arithmetic. For example, our definition of subtraction really used an open number sentence. "7 - 5 is that number which makes the open sentence 5 + □ = 7 a true statement."

Problem*

1. Tell which of the following sentences is an open sentence. If possible, tell whether the sentence is true or false.
   a. 5 + 2 = 9 - 2
   b. 5 + 3 = 9 - 3
   c. 6 + □ = 9 - □
   d. 3 is a factor of 15
   e. 7 - 2 = 
   f. V = 3
   g. IV = 5 - 1
   h. D = 1000 ÷ 2
   i. 

Any number sentence has to have a verb. The most common ones are: "is equal to;" "is not equal to;" "is more than;" "is less than;" "is more than or is equal to;" "is less than or is equal to." The symbols which we use for these verbs are listed below.

= ; "is equal to" ; 3 + 4 = 7
≠ ; "is not equal to" ; 5 - 2 ≠ 4
> ; "is more than" ; 7 - 3 > 1
< ; "is less than" ; 5 < 10
≥ ; "is more than or equal to" ; 9 ≥ any one-digit number
≤ ; "is less than or equal to" ; 0 ≤ any whole number

None of the examples listed above are open sentences. They make statements about specific numbers which are described or represented by a single numeral such as 7 or by a mathematical or number phrase such as 3 + 4. If we want to write an open number sentence, we will use an open number phrase such as □ + 7 or 17 - □ where the symbol □ is used to help you remember that the empty space may be filled by some numeral. Because symbols like □ are awkward to type or write, we frequently use a letter such as n or a for the same purpose. Thus, a simple open number phrase may be written as $n + 7$ instead of $\square + 7$ and an open number sentence as $n + 7 = 10$. What number or numbers will now make this open sentence a true statement? In this case the answer is easily obtained by trial. $3 + 7 = 10$, while $0 + 7 \neq 10$, $1 + 7 \neq 10$, $2 + 7 \neq 10$, $4 + 7 \neq 10$ and we see that 3 is the only number which does the trick.

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* Solutions to problems in this chapter are on page 137.

What number or numbers will make the open sentence $\square < 5$ a true statement? Again, by trial we find that $0 < 5$, $1 < 5$, $2 < 5$, $3 < 5$ and $4 < 5$ are true statements while $5 < 5$, $6 < 5$, $7 < 5$, etc., are false statements. Thus we see that any member of the set $\{0, 1, 2, 3, 4\}$ makes the statement true. What about the open sentence $n + 6 < 11$? We can translate the sentence into words by saying "the sum of a certain number and 6 is less than 11" and we see the numbers which make this a true statement are again the members of the set $\{0, 1, 2, 3, 4\}$.

Problem

2. Tell which of the following sentences is an open sentence. If possible, tell whether it is true or false. If it is an open sentence, tell what number or numbers will make it a true sentence.
   - a. $DC = 600$
   - b. $CD = DC$
   - c. $a = a$
   - d. $a + b = b + a$
   - e. $5 + a = 5$
   - f. $a = 5$
   - g. 3 is a factor of n
   - h. 3 is a factor of $3 \times n$.

Solving Open Sentences

Open number sentences are called equations if the verb in them is "=". Sentences with any of the other verbs listed above are called "inequalities." Those numbers which make the sentences true are called solutions of the equations or inequalities. When you have found the entire set of solutions of an open sentence, you can say that you have solved the sentence.

Open number sentences are frequently used to solve problems. To do this, you must be able to describe the numbers in the problems by number phrases and to translate the clues given in the problem into an equation or inequality. To work with number phrases you must be able to translate the phrase into words. The open phrase $\square + 5$ or the equivalent phrase $n + 5$ may be translated as "a number increased by 5." It may, of course, have many different translations, such as:

"a number $\underline{n}$ added to 5;"
or "the sum of a number and 5;"
or "five more than a number $\underline{n}$."

However, all of the translations have the same mathematical meaning. Furthermore, all the English translations mean the same as "$n + 5$." With practice, we learn to understand the different ways of expressing a number phrase.

Consider the following sentences:

The sum of 8 and 7 is 19;
$n + 8 = 16$;
$4 + 5 = 10 - 1$;
$3 < 2 + 6$;
$5 > 2 + \square$;
$3 + n = n + 3$.

These are all number sentences. One of them is false, two of them are true and three of them are open sentences. One of the open sentences is true for only one number, one of them is true for three different numbers, and one of them is true for any whole number. Can you identify each statement with the appropriate open sentence?

Examination of the sentences in the list should verify that $n + 8 = 16$ is true for only one number, $5 > 2 + \square$ is true for three different numbers, and $3 + n = n + 3$ is true for any whole number.

Since open sentences contain words or symbols which do not refer to only one thing, frequently they are neither true nor false. However, this is not always the case. It is interesting to examine the following sentences. What can you say about their truth?

"$13 - x = 7$."
"George was the first President of the United States."
"$3 + x = x + 3$."
"If Jimmy was at Camp Holly all day yesterday,
then he was not at home at that time."

These sentences are similar in that each contains a word or symbol which can refer to any one of many objects. Do you see any difference between the first two sentences and the second two? Can the first two sentences be true? Can the first two sentences be false? Can either of the last two sentences ever be false? We can see that the first two may be false, whereas the last two are always true.

We said above that by *solving* an open equation or inequality we mean finding that number, or all those numbers which make the sentence a true one. At this time, you can do this primarily by trial and error after thinking carefully about what the sentence says. For instance, to solve the equation $n - 4 = 7$ means to find the number which is the result of adding 4 to 7. The answer is, of course, 11. To solve $n + 5 = 32$ is to find that number which added to 5 will give 32, or $n = 32 - 5 = 27$. On the other hand, to solve $n - 4 \leq 7$ means to find all those numbers from which 4 may be subtracted and for which the result will be less than or equal to 7. Is 2 such a number? No, because $2 - 4$ is not a whole number. Is 3? No. But 4, 5, 6, 7, 8, 9, 10 and 11 do make the sentence true. On the other hand, $12 - 4 = 8$ which is more than 7. So 12 and any other larger number make the sentence false. We see that the set of solutions is $\{4, 5, 6, 7, 8, 9, 10, 11\}$.

Use of Mathematical Sentences

The use of a mathematical sentence to solve a problem may be illustrated as follows:

There are 22 children in a class. 10 of the children are boys. How many are girls? We can write several different open sentences to express the relationship among the numbers involved. Thus, $10 + n = 22$ or $22 - n = 10$. In each case, we can think "a number added to 10 gives the sum 22." The only number which makes this a true statement is 12. This is the solution of the number sentence, and the answer to the problem is "there are 12 girls in the class."

In using number sentences to solve problems, the key to the situation is in recognizing the relationship between the numbers in the problem. This relationship is written as a number sentence. The solution of the number sentence is found and the result used to answer the question posed by the problem. One more example.

John put 23 of his marbles in a bag and Jim put 48 of his marbles in the same bag. If Tom takes out 35 marbles, how many are left in the bag? The number relationship can be thought of as "the number of marbles left in the bag plus the number Tom took out equals the number John and Jim put in." This gives the equation $n + 35 = 23 + 48$. Someone else might think of the relationship as "the number of marbles left in the bag is the difference between the number John and Jim put in and the number Tom took out."

This yields $n = (23 + 48) - 35$. Of course, both open sentences have the same solution: 36. The answer to the problem is: "There are 36 marbles left in the bag." Notice that $n + 35 = 23 + 48$ if and only if $n = (23 + 48) - 35$.

Solution Set on the Number Line

Frequently, a picture of a solution set using the number line can be drawn. Consider the following example for the open sentence

$$\square + 3 = 8.$$

This open sentence has the solution, 5. The solution set is $\{5\}$. On the number line this solution can be represented as shown below:

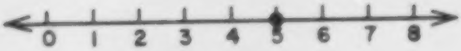


Figure 12-1. $\square + 3 = 8$.

Since the only solution for the sentence is 5, a solid "dot" or circle, is marked on the number line to correspond with the point for 5. No other mark is put on the drawing.

The solution set of the inequality $n - 4 \le 7$ which we solved previously can be represented thus:

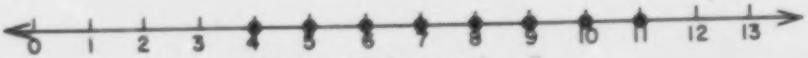


Figure 12-2. $n - 4 \le 7$.

Note that on the number line we indicate the solution set by heavy solid dots. The solution set of $n - 4 > 7$ cannot be completely represented because it consists of all numbers greater than 11. We can indicate it, however, as in Figure 12-3 where the heavy dots continue right up to the arrow and the word "incomplete" indicates that all the numbers represented by points still further to the right belong in the solution.

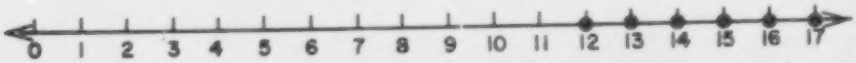


Figure 12-3. $n - 4 > 7$.

Operations on the Number Lines

Number sentences can also be pictured on the number line as shown below.

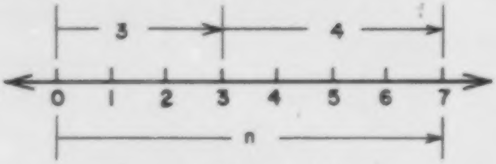


Figure 12-4. $3 + 4 = n$.

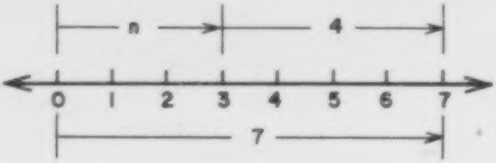


Figure 12-5. $n + 4 = 7$.

Recalling that $n + 4 = 4 + n$, and that $4 + n = 7$ if and only if $n = 7 - 4$, we can observe that

$$n + 4 = 7, \quad 4 + n = 7, \quad \text{and} \quad n = 7 - 4$$

are all statements which say the same thing. We can picture the number sentence $n = 7 - 4$ as follows and note that the arrow for n agrees with the arrow for n in the picture for $4 + n = 7$ in Figure 12-7.

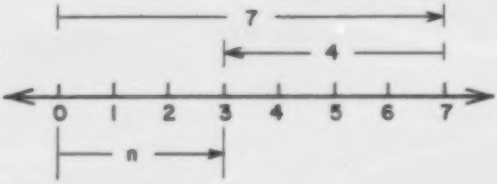


Figure 12-6. $n = 7 - 4$.

The picture for the sentence $4 + n = 7$ takes the following form:

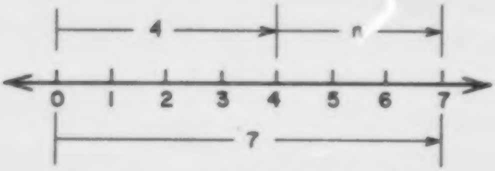


Figure 12-7. $4 + n = 7$.

Just as addition and subtraction may be shown on the number line, multiplication and division may also be illustrated. For example, to show 3×4, consider an arrow for 4. Three such arrows laid end-to-end (tail to head) indicate 3×4 (see Figure 12-8).

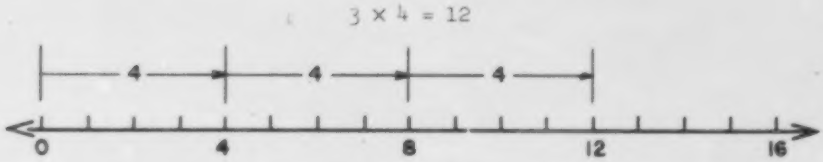


Figure 12-8. Multiplication on the number line.

Figure 12-9 illustrates the operation of division for $12 \div 4$; exactly three 4-arrows fit end-to-end, showing $12 \div 4 = 3$.

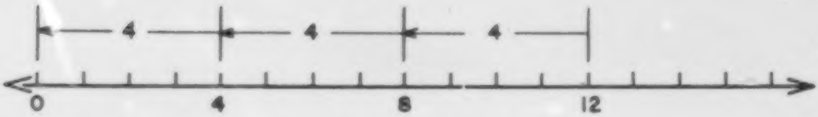


Figure 12-9. Division on the number line.

For the division process indicated by $17 \div 3$, we see from Figure 12-10 that the algorithm will yield the quotient 5 and the remainder 2.

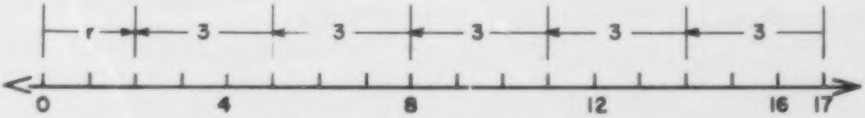
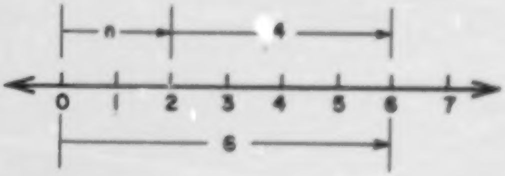


Figure 12-10. $17 = (5 \times 3) + 2$.

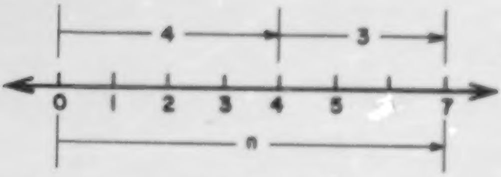
Exercises - Chapter 12

1. Write a number sentence shown by each number line.

   a.



   b.



2. What number must $\underline{p}$ represent so each mathematical sentence is true?

   a. $10 + p = 30$

   b. $0 = p + 0$

   c. $0 - p = 0$

   d. $p - 9 = 20$

   e. $40 - p = 10$

   f. $15 - p = 12$

3. Write a mathematical sentence using $\underline{n}$, 12 and 15.

4. Tell what operation is used to find $\underline{n}$ in each of these true mathematical sentences.

   a. $5 + 6 = n$

   b. $n = 7 - 4$

   c. $n + 2 = 43$

   d. $75 = n + 31$

   e. $5 + n = 6$

   f. $91 - 60 = n$

5. Write $<$, $>$, or $=$ in each blank so each mathematical sentence is true.

   a. 8 ____ 6

   b. $3 + 4$ ____ 6

   c. $(20 + 30)$ ____ $(30 + 20)$

   d. $(200 + 800)$ ____ $(200 + 700)$

   e. $(1200 + 1000)$ ____ $(1000 + 1200)$

6. How much of a number line must be shown to picture these mathematical sentences?

   a. $15 + 18 = n$

   b. $140 - n = 40$

   c. $n = 10 + 20 + 30$

7. Apply the "undoing" idea to these mathematical sentences. Solutions to (a) and (b) are given.

	Do	Undo
a.	$5 + 2 = 7$	$7 - 2 = 5$
b.	$6 - 4 = 2$	$2 + 4 = 6$
c.	$5 + 3 = 8$	
d.	$18 - 10 = 8$	
e.	$25 + 20 = 45$	
f.	$3 + n = 6$	
g.	$n - 2 = 4$	
h.	$p - n = q$	

8. The mathematical sentence $3 \times 4 = n$ is shown on the number line in Figure 12-11; write the mathematical sentence for Figure 12-12.

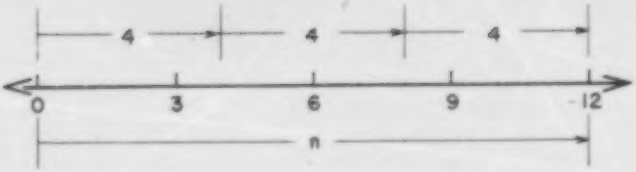


Figure 12-11. The sentence $3 \times 4 = n$.

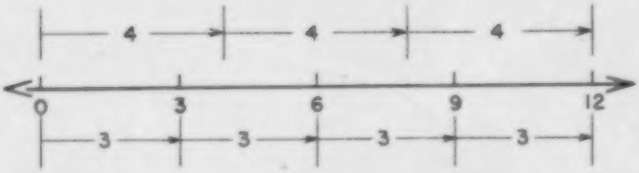


Figure 12-12.

9. Draw on the number line the algorithm indicated by $18 \div 9$.

Solutions for Problems

1. a. Not open; true
   b. Not open; false
   c. Open
   d. Not open; true
   e. Not open; true
   f. Not open if V stands for Roman numeral; false. If V stands for some unknown number, then open.
   g. Not open; true
   h. Not open; true
   i. Not open; true

2. a. Not open; true
   b. Not open; false
   c. Open; true for all numbers *a*
   d. Open; true for all numbers *a* and *b*
   e. Open; 0
   f. Open; 5
   g. Open; true for all multiples of 3
   h. Open; true for all numbers *n*

Chapter 13

POINTS, LINES AND PLANES

Introduction

Up to this point in our work we have been studying whole numbers and their properties. However, numbers are not the only things in mathematics that interest people. Points, lines, curves, planes and space also belong in mathematics. The study of such ideas is called Geometry. For over 4000 years men have studied geometry, trying to understand better the world in which they live.

Sometimes geometry has been closely associated with the process of measurement. Students have computed perimeters, areas and volumes associated with certain geometric figures. Sometimes geometry is treated as a deductive science, building a whole series of theorems on the basis of a few undefined terms and unproved axioms.

Neither of these is the approach taken in the next three chapters. Rather, the object is to direct attention to some of the geometric properties of familiar objects which do not depend on measurement and to do so without setting up a formal deductive system. However, thinking logically about the ideas presented is important, as logical thinking is important in understanding any mathematics at any level. In Chapter 16 the ideas and processes of measurement will be introduced.

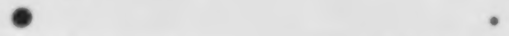
We begin with the consideration of such mathematical ideas as point, line, plane, space, curve and simple closed curve. Then some representations of these ideas in the physical world will be considered in order to use these concepts and their representations to improve our understanding of the world in which we live.

Points

What is meant by words such as point, line, plane, curve and space? If we look up the meaning of any word in a dictionary, we will find the word defined in terms of other words. If we continue looking up the words used in the definition, we eventually will find one of these words defined in terms of the original word whose meaning we were seeking or some succeeding word in the chain. Hence every such set of definitions is circular. In order for the dictionary to be helpful, we must know the meaning of some word in the circle prior to using the dictionary.

In geometry, no attempt is made to give a definition for such terms as "point" or "line." What is done instead, is to describe many properties of points and lines. This will help us in thinking of physical models of geometric figures. We will draw pictures of these physical models, but it should be remembered that the pictures are only a help in thinking about the geometric ideas and are not the objects being discussed.

Thus, the idea of a point in geometry is suggested by the tip of a pencil or a dot on a piece of paper. A dot represents a point in that it indicates a location at least approximately. A point might be described as an exact or precise location in space. But note that even this description involves the word "space" and when we try to describe what is meant by "space," we will do it in terms of "point." A point might also be thought of as represented by a corner of a room where two walls and the ceiling meet, or as the end of a sharply pointed object. These are representations in the following sense. The dot made on a sheet of paper is merely an attempt to mark the idealized geometric entity, the "point." In fact, the dot covers not one point but an infinite number (in this instance more than can be counted). Viewed under some magnifying device such as a microscope, any dot is clearly seen to cover many locations. Hence no device can be used to mark a point accurately.



A poor representation of a point.

A better representation of a point.

Figure 13-1.

Also, we observe that a point may be thought of as a fixed location. A point does not move. If the dot made on a sheet of paper were erased, the location previously marked by this dot still would remain. Again, if the sheet of paper were moved to some other place, the point originally marked by the dot would remain fixed. Perhaps a more graphic demonstration of the permanency of the geometric point is given by the demolition of a building. The points occupied by each corner of each room remain unchanged. The difference is that they are no longer represented by the physical objects called the corners. They would now have to be described by some set of directions leading to the location, such as 10 feet north of some marked point and then 12 feet up. Finally, think of a pencil held in some position. Its tip represents a geometrical point. If the pencil is moved, its tip now represents a different geometrical point.

A point is usually represented by a dot. It is customary to assign a letter to a point and thereafter to say "the point A" or "the point B." We write the letter we have assigned to a particular point next to the dot which represents the point.

Problems*

1. Look up the definition of "point" in your dictionary. Look up the key words in this definition and continue until you see a circular pattern. How many different words occur in the circle?
2. Do the same for "small."

Sets of Points

Once the geometrical meaning of point is understood, we are then prepared to envision geometrical space or simply space.

Space is the set of all points.

Since points can be thought of as represented by locations, space may be thought of as represented by all possible locations.

If we take two points A and B in space and think of moving the tip of a pencil from the location of A to the location of B, the path traced by the pencil tip will give a pretty good idea of a curve from A to B. There are, of course, many paths which might have been taken besides the one we did select. Each path consists of a set of points and the set of all points on such a path is called a curve.

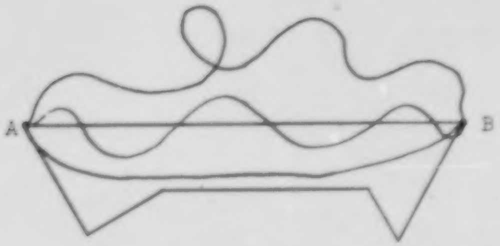


Figure 13-2. Five curves from A to B.

Each of the five paths represented above is called a curve. Even the one usually called a straight line is a special case of a curve. (Note that this mathematical use of the word "curve" is not the same as the ordinary use.)

A curve is a set of points: all those points which lie on a particular path from A to B.

---

* Solutions for the problems in this chapter are on page 150.

Now think of the particular curve which may be *represented* by a string tightly stretched between A and B. This special curve is called a *line segment* or a *straight line segment*.

Another representation of a line segment would be the pencil mark drawn with a ruler and pencil connecting two points. The curve includes the points A and B which are called the endpoints of the line segment. The line segment can be thought of as the line of sight between A and B and can be described as the most direct path. The line segment exists, of course, independently of any of its representations. For example, if the stretched string were removed, the line segment would remain since it is a set of locations. The symbol for the line segment determined by the points A and B is $\overline{AB}$. The fact that we say "*the* line segment $\overline{AB}$" implies that there is only one such segment.

> If $\overline{AB}$ is extended in both directions along the line of sight so that it does not stop at any point, the result is a *straight line*. Its symbol is $\overleftrightarrow{AB}$.

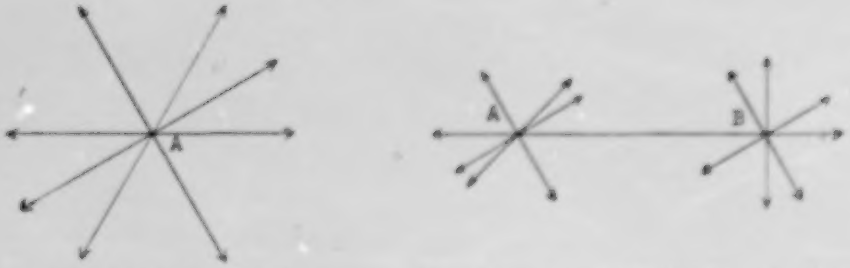
For brevity a straight line is called simply a line. Note that a line is a set of points, a particular set of points whose properties we describe but which cannot wholly be represented in a figure because of its indefinite length. We have to use our imagination to conceive of the unlimited nature of the line.

Problems

3. Mark three points A, B and C on a sheet of paper. Draw all possible line segments determined by these points. How many segments did you get? Name these segments. How many lines are determined? Did you take care of all possible situations?
4. Mark four points A, B, C and D on a sheet of paper. Continue as in Problem 3.

Properties of Lines

From the representation of a line segment it is possible to abstract certain properties of the line. Thus: through a point A many lines can be drawn; in fact so many that we cannot possibly count them all. In other words, there may be infinitely many lines passing through a single point A.



Many lines through A. One line through both A and B.

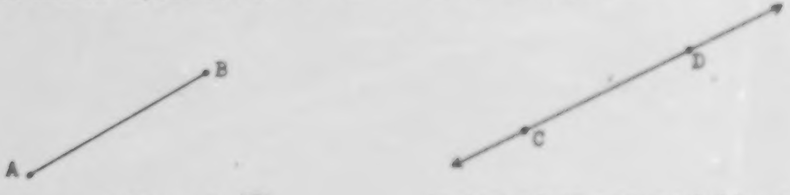
Figure 13-3.

If a second point B is given different from A, there is always one line which passes through both A and B but more importantly, there is only one such line, which is said to be determined by the two points. The intersection of two lines is, of course, the set of all those points which belong to both lines. There is usually just one such point. These two important properties of lines can be stated as follows:

1) If two distinct lines are given which intersect, their intersection consists of exactly one point.
2) If two distinct points are given, there is exactly one line which contains both points.

In the representation of points as dots, it may be possible to draw two or more distinct lines between these dots if the dots are not initially small enough. Thus, . However, the realization that the dots are only an approximation of the idealized notions of point and line should clarify this apparent contradiction. The use of smaller and smaller dots quickly forces the correct conclusion.

The line determined by two different points such as C and D in Figure 13-4 is called "the line CD." The symbol for it is $\overleftrightarrow{CD}$. Note that $\overline{CD}$ is the symbol for the line segment with endpoints at C and D while $\overleftrightarrow{CD}$ is the symbol for the line determined by C and D but which goes on and on beyond either C or D in both directions.

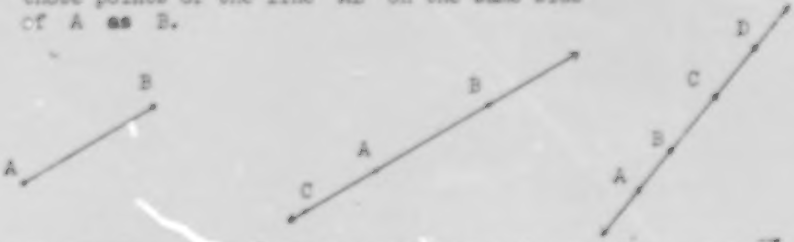


A representation of $\overline{AB}$ An incomplete representation of $\overleftrightarrow{CD}$.

Figure 13-4.

Besides $\overline{AB}$ and $\overleftrightarrow{AB}$ the two points A and B determine another particular set of points called "tho ray AB" and for which the symbol is $\overrightarrow{AB}$.

> The ray $\overrightarrow{AB}$ consists of the point A; and all those points of the line $\overleftrightarrow{AB}$ on the same side of A as B.



A representation of $\overline{AB}$.

(a)

Incomplete representations of $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

(b)

Many rays $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CA}$, $\overrightarrow{DC}$ determined by points in one line.

(c)

Figure 13-5.

A beam of light emanating from a pinpoint source is another excellent representation of a ray.

Since there are only two directions in a line from a fixed point on the line, there can be only two distinct rays on the line with the fixed point as a common endpoint. See Figure 13-5b. However, since any point of the line may serve as the endpoint of a ray on the line, a line contains more rays than can be counted. See Figure 13-5c.

A ray may be a less familiar concept than a point or a line, but it is a very useful one, particularly when we come to talk about angles.

Problems

5. Mark three points A, B and C in that order on the same line. Indicate $\overrightarrow{AB}$, $\overrightarrow{BA}$, $\overrightarrow{BC}$, $\overrightarrow{AC}$, $\overrightarrow{CA}$, $\overrightarrow{CB}$. Which of these are names for the same ray?
6. What is the difference between $\overline{AB}$ and $\overleftrightarrow{AB}$; between $\overline{AB}$ and $\overrightarrow{AB}$; between $\overleftrightarrow{AB}$ and $\overrightarrow{AB}$?
7. Make a point A on a sheet of paper. Draw four rays with endpoint A. How many rays are possible all with endpoint A?
8. Draw a ray $\overrightarrow{AB}$. How many rays are there with endpoint A which contain point B? Is $\overline{AB}$ contained in $\overrightarrow{AB}$? How many line segments are there on $\overrightarrow{AB}$ which have A as one endpoint?

Planes

Perhaps a more familiar concept is that of the set of points we call a plane. Once again we do not define a plane, but we describe its properties and its relationship to lines and points and thus attempt to get a good idea of what is meant by it. Any flat surface such as the wall of a room, the floor, the top of a desk or a door in any position suggests the idea of a part of a plane in mathematics. But as with a line, a plane is thought of as being unlimited in extent. A plane is represented by a picture or a drawing of a flat surface, but this is only a representation of a part of a plane. An ever-growing table top provides a better and better representation. Also, we must remind ourselves that we are only representing certain sets of points in space. If the table were removed, the set of points (locations) does not change.

A plane of which the table top is a partial representation is the set of all points of all the lines obtained by extending the line segments with endpoints in the table top. In this way it is clear that a plane contains more lines than can be counted, in other words, infinitely many lines. Moreover, if two points of a line are contained in a plane, then the entire line is contained in the plane.

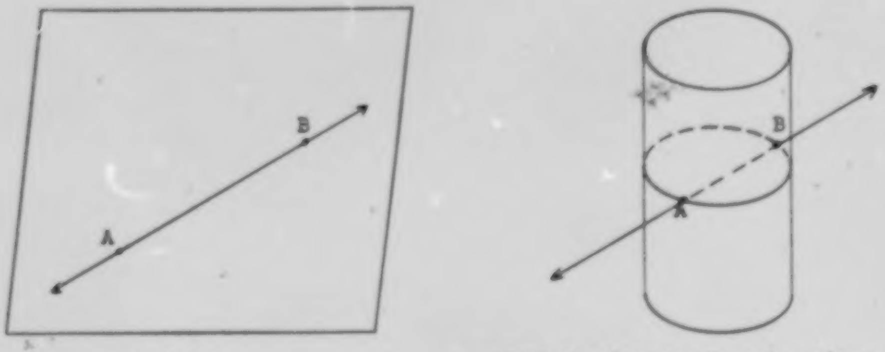
The Relationships of Points, Lines and Planes

Consider now the relationships between points, lines and planes. We shall try to put them down in a somewhat systematic fashion. Remember that space is simply the set of all points and that lines and planes are special sets of points some of whose properties we have tried to specify. We have already noted certain relationships, one of them being:

Property 1. Through any two different points there is exactly one line.

We think of a plane as being "flat." If any two points in the plane are selected, they determine a line. Where are all the other points of this line? The straightness of the line and the flatness of the plane suggest that they lie in the plane. Thus:

Property 2. If two different points lie in a plane, the line determined by the points lies in the plane.



A plane.

Surface of a can is not a plane.

Figure 13-6.

In Figure 13-6 the surface of a tin can is not a plane since $\overleftrightarrow{AB}$ cuts through the space inside the can and does not lie wholly on the surface.

Can there be more than one plane containing two different points? Think of two points at the hinges of a door. The door, which represents part of a plane, can swing freely, and therefore there must be many planes through the two given points.

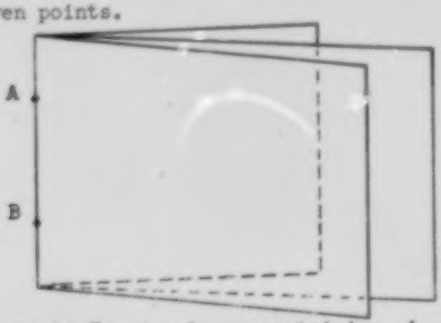


Figure 13-7. 3 planes containing A and B.

Another way to think of it would be to think of a 3 × 5 card held at opposite corners between the thumb and third finger. The card can spin freely and in each position represents a portion of a plane.

Property 3. Through 2 points in space, and hence through a line in space, there are many possible planes.

It seems that two different points determine a line. How many are needed to determine a plane? By "determine" we mean that there is at least one plane containing the given points and no more than one.

The card mentioned above held between thumb and third finger can no longer spin if a third finger is extended. The tips of the three fingers determine a plane. A three-legged stool always sets firmly on the floor while a badly made four-legged table may wobble. These illustrate the idea that three points can fix a plane. Of course, they must not lie on the same line since, for instance, if there were another point C on $\overline{AB}$ in Figure 13-7, all three planes would contain it as well as A and B. So:

Property 4. Any three points not in the same line determine one and only one plane.

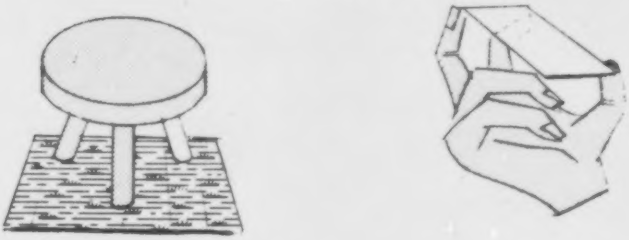


Figure 13-8. A plane is determined by 3 points not in a line.

Two intersecting lines also determine a plane since we can pick two points in one line and the third one in the other line and use Property 4. In the same way a line and a point which is not in it determine one and only one plane.

Problems

9. Do three points always determine a plane? If not, why not?
10. May four different points no three of which lie in the same straight line lie in one plane? Must they lie in one plane? Draw pictures to support your answer.

Intersections of Lines and Planes

If the intersections of lines and planes are considered, three more interesting properties can be noted.

Two different lines may not intersect at all, but if they do, what could their intersection be? A single point. Just think of two pencils and how they may be held.

If a line and a plane intersect, their intersection must be only a single point unless the whole line lies in the plane. In this case the line is a subset of the plane.

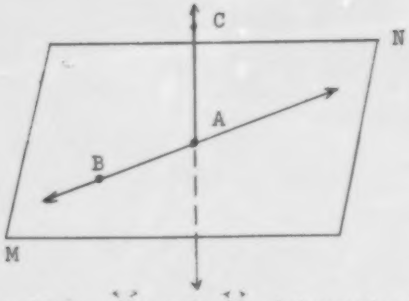


Figure 13-9. $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ intersect at A.
Line $\overleftrightarrow{AB}$ lies in plane MN.
$\overleftrightarrow{AC}$ intersects plane MN at A.

These ideas are summarized in the following properties:

Property 5. If two different lines in space intersect, their intersection is one point.

Property 6. If a line and a plane intersect, their intersection is either one point or the entire line.

A more difficult thing to see, perhaps, is how two planes intersect. Of course, the floor and ceiling of an ordinary room represent portions of two planes which would not intersect at all. The floor and a flat wall seem to intersect in a straight line. In fact, consideration of any two planes which have some points in common will probably indicate that they must meet in a straight line. This is indeed true. It is stated in Property 7.

Property 7. If two different planes intersect, their intersection is a straight line.

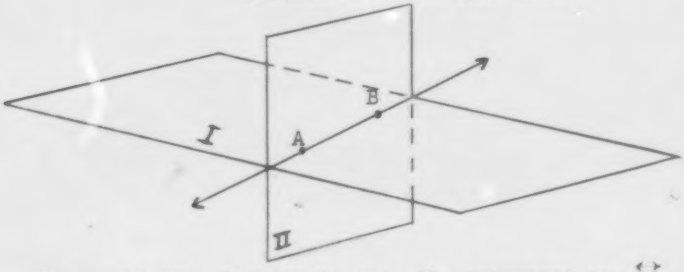


Figure 13-10. Planes I and II intersect in $\overleftrightarrow{AB}$.

The ideas of point, line segment, line, ray, plane and space which have been considered here are basic to geometry, but, obviously, there are many other interesting point sets to look at. This we will begin to do in the next chapter.

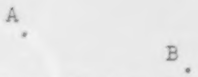
Exercises - Chapter 13

1. How many different lines may contain:
    - a. one certain point?
    - b. a certain pair of points?

2. How many different planes may contain:
    - a. one certain point?
    - b. a certain pair of points?
    - c. a certain set of three points not all in the same line?

3. a. If two different lines intersect, how many points are there in the intersection?
    - b. If two different planes intersect, how many lines are there in the intersection? How many points?

4. Given two points A and B as shown below:



   - a. How many line segments can you draw with endpoints A and B ?
   - b. How many lines are there that contain both A and B ?

5. Draw points A and B.
    - a. Draw a ray with endpoint A.
    - b. Draw a ray with endpoint B.
    - c. Draw $\overline{AB}$.
    - d. Draw $\overrightarrow{AB}$.

Which of the following is correct?

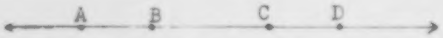
6. A line has
    - a. one endpoint.
    - b. two endpoints.
    - c. no endpoints.

7. A ray has
    - a. one endpoint.
    - b. two endpoints.
    - c. no endpoints.

8. A line segment has
   a. one endpoint.
   b. two endpoints.
   c. no endpoints.

9. Two points in space are contained in
   a. only one plane.
   b. many, many planes, but we could count them.
   c. more planes than can be counted.

10. Complete this sentence: Two intersecting planes in space intersect in a __________________ .

11. Consider 

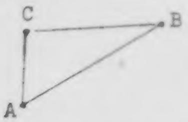
   a. What is the union of $\overline{AB}$ and $\overrightarrow{BC}$?
   b. What is the union of $\overline{AB}$ and $\overrightarrow{CB}$?
   c. What is the union of $\overline{AC}$ and $\overline{CD}$?
   d. What is the union of $\overrightarrow{BC}$ and $\overrightarrow{BA}$?

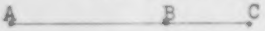
12. How many lines can be drawn through four points, a pair of them at a time, if the points lie:
   a. in the same plane?
   b. not in the same plane?

Solutions for Problems

1. The answer will depend on your dictionary. In one dictionary the definition of "point" involves "line" and the definition of "line" involves "point."

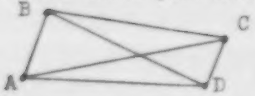
2. "Small" in one dictionary is defined in terms of "little" and "little" in terms of "small."

3. 

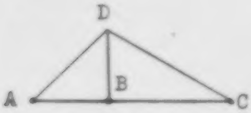
In this case there are determined three segments $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$, and also three lines $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$ and $\overleftrightarrow{AC}$.
But in the other case 

, while there are still the same three segments, there is now only one line $\overleftrightarrow{AC}$.

4. There are three possible cases.



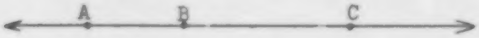
In this case there are six segments $\overline{AB}$, $\overline{AC}$, $\overline{AD}$. $\overline{BC}$, $\overline{BD}$ and $\overline{CD}$ and also six lines.

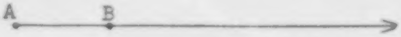


In this case there are still the same six segments, but only 4 lines: $\overleftrightarrow{AD}$, $\overleftrightarrow{BD}$, $\overleftrightarrow{CD}$ and $\overleftrightarrow{AC}$.

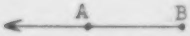
Again 6 segments, but now only one line $\overleftrightarrow{AD}$.

5.

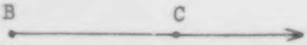




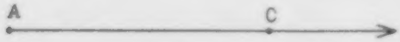
$\overrightarrow{AB}$



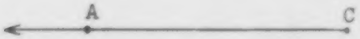
$\overrightarrow{BA}$



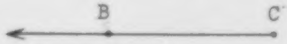
$\overrightarrow{BC}$



$\overrightarrow{AC}$



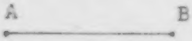
$\overrightarrow{CA}$

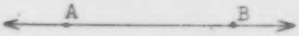


$\overrightarrow{CB}$

$\overrightarrow{AB}$ and $\overrightarrow{AC}$ are names for the same ray.

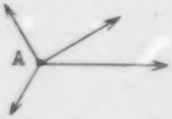
$\overrightarrow{CA}$ and $\overrightarrow{CB}$ are names for the same ray.

6. $\overline{AB}$ is the segment 

$\overleftrightarrow{AB}$ is the line 

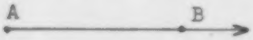
$\overrightarrow{AB}$ is the ray

7.



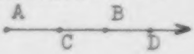
There are infinitely many rays possible with endpoint A.

8.



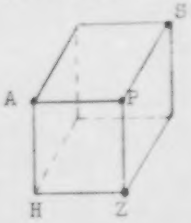
One ray from A contains B.

Yes. $\overline{AB}$ is contained in $\overrightarrow{AB}$

Infinitely many  such as $\overline{AC}$, $\overline{AD}$, etc.

9. No, as they may all lie in one line.

10. They may but they do not have to. In a picture of a cubical block of wood, points A, P, H and Z lie in one plane, but points A, P, S and Z do not.



Chapter 14

CLOSED CURVES, POLYGONS AND ANGLES

Intersecting Planes and Lines

In the last chapter we considered lines and planes and some of their properties and relationships. In particular, we looked at the various ways in which these sets of points could intersect each other, saying that

1. If two different planes intersect, the intersection is a single line.
2. If a line and a plane intersect, the intersection is either a single point or the entire line.
3. If two different lines intersect, the intersection is a single point.

It is possible, of course, for two planes not to intersect at all, in which case we say that they are parallel. The same is true for a line and a plane or for two lines. See Figure 14-1.

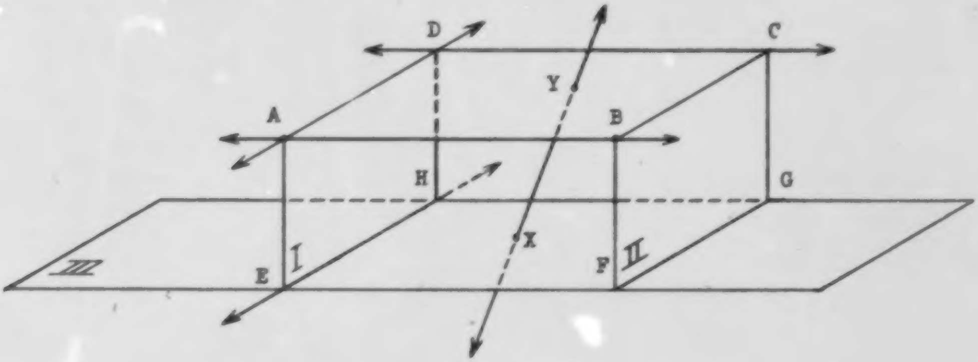
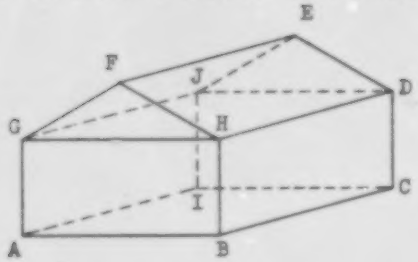


Figure 14-1.

In this figure, planes I and II are parallel whereas planes I and III intersect in $\overleftrightarrow{EH}$. $\overleftrightarrow{AB}$ intersects plane I in point A but $\overleftrightarrow{AB}$ is parallel to plane III. Two planes always either intersect or are parallel. It is possible, however, for two lines not to intersect and yet not to be parallel to each other either. This is the case with $\overleftrightarrow{XY}$ and $\overleftrightarrow{EF}$. Such lines are said to be skew and they never lie in the same plane. We speak of two lines as being parallel only if they lie in the same plane and do not intersect. $\overleftrightarrow{AD}$ and $\overleftrightarrow{EH}$ are parallel.

Problem *

1. Consider this sketch of the outline of a house.



Think of the lines and planes suggested by the figure. Name lines by a pair of points and planes by three points. Name:

   a. a pair of parallel planes
   b. a pair of planes whose intersection is a line
   c. three planes that intersect in a point
   d. three planes that intersect in a line
   e. a line and a plane which do not intersect
   f. a pair of parallel lines
   g. a pair of skew lines
   h. three lines that intersect in a point

The Separation Properties of Points, Lines and Planes

There is another very important idea closely connected with the relationship between points, lines and planes, and that is the idea of separation. What is involved may be made clear by some examples and illustrations. Think of a plane represented by the wall of a room. This plane separates space into two sets of points, those in front of the wall and those behind it. In this sense we think of any plane as having two sides and we say that a plane separates space, that is, it divides the points of space into three subsets, one of which is the plane itself while the other two are the points on each side of the plane. By convention we refer to these latter two sets of points as half-spaces. Note that the separating plane does not lie in either half-space. Later on we shall use "half-plane" and "half-line" in a similar sense. Points are in the same half-space if they are on the same side of the plane. If points such as A and B in Figure 14-2 are on the same side of the plane, then there always exist curves connecting A and B which do not intersect the plane. In particular $\overline{AB}$ does not intersect the plane. On the other hand, if points such as A and C are on opposite sides of the plane, then any curve connecting A and C even $\overline{AC}$ must intersect the plane. Note that while the segment $\overline{AB}$ does not intersect the separating plane in Figure 14-2, the line $\overleftrightarrow{AB}$ may very well do so.

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* Solutions for problems in this chapter are on page 167.

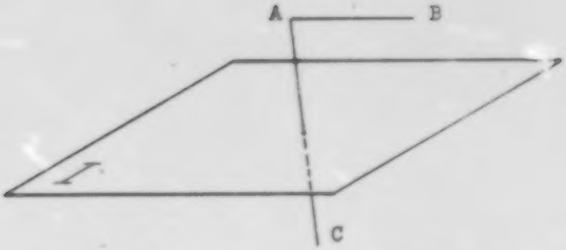


Figure 14-2. Plane I separates space.

In the same way, if we consider a certain plane and a line in that plane, the line separates the points of the plane into two half-planes. Thus, in Figure 14-3, $\overleftrightarrow{MN}$ separates plane I into two half-planes such that A and B lie in the same half-plane but A and C lie in different

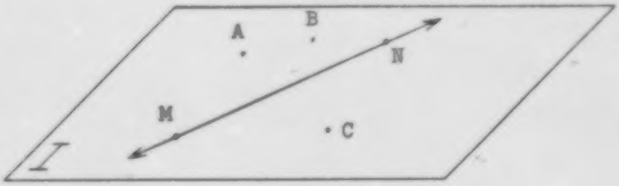


Figure 14-3. A line separates a plane.

half-planes. The line $\overleftrightarrow{MN}$ does not lie in either half-plane. There exist curves in plane I connecting A and B which do not intersect $\overleftrightarrow{MN}$. $\overline{AB}$ is such a curve. On the other hand, every curve in plane I connecting A and C must intersect $\overleftrightarrow{MN}$. We see that $\overline{AC}$ does.

In the same manner, a point in a line separates a line. In

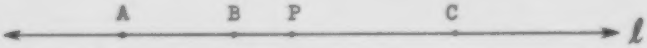


Figure 14-4. A point separates a line.

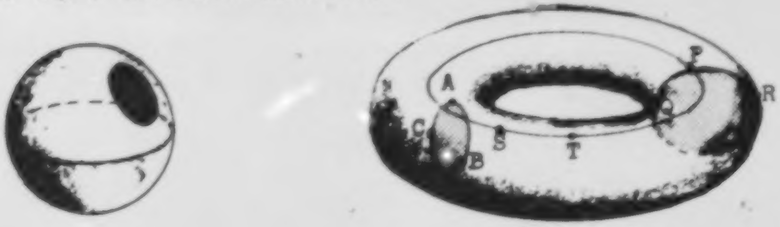
Figure 14-4, P separates line ℓ into two half-lines such that A and B lie in the same half-line, but A and C lie in different half-lines. The point P does not lie in either half-line. Again $\overline{AB}$ does not intersect the separating point P while $\overline{AC}$ does.

The three cases are much the same idea applied in different situations. Thus:

1. Any plane separates space into two half-spaces.
2. Any line of a plane separates the plane into two half-planes.
3. Any point of a line separates the line into two half-lines.

The separation properties of
points with respect to lines,
lines with respect to planes, and
planes with respect to space

have not usually had much attention drawn to them in elementary geometry, but they really are quite interesting and important although a little tricky to comprehend at first acquaintance. It might be of interest to consider whether there are any curves which are not separated by points or any surfaces which are not separated by curves. Does a single point separate a circle? Does a circle separate a plane? Does a circle separate the surface of a sphere? Does a circle always separate the surface of an inner tube? See Figure 14-5. How about circle ABC ? How about circle PQR ? How about circle PST ?



A circle on a sphere. Circles on an inner tube.

Figure 14-5.

A further discussion of some of these questions will be found later on in this chapter. Some experimentation will probably convince you that a circle always does separate the surface of a sphere but does not always separate the surface of an inner tube. Neither circle PQR nor circle PST separates that surface.

Problems

2. Consider a piece of paper as a representation of a plane.
   a. Does a segment separate the plane? A ray? A line?
   b. Into how many parts do two intersecting lines separate a plane? Two parallel lines?

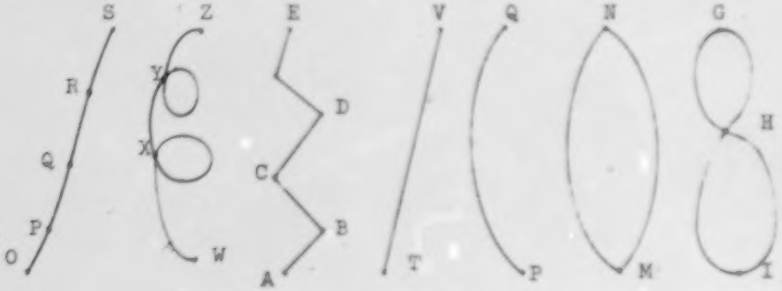
3. a. Does a half-plane separate space? Does a line?
   b. Into how many parts do two intersecting planes separate space?

Plane Curves

Let us go back to the idea of a curve and see what more we can learn from it. A curve from A to B is any set of points which can be represented by a pencil point which started at A and moved around to end up at B. Such a curve might wander around in space and cross and recross itself many times. We want to confine our attention to fairly simple curves and so we impose some restrictions. The first is that our pencil point stay in the same plane. Such curves are called plane curves. We can represent them by figures we draw on a sheet of paper.

> A plane curve is a set of points which can be represented by a pencil drawing made without lifting the pencil off the paper.

Line segments are examples of such curves. Curves may or may not contain portions that are straight. In everyday language we use the term "curve" in this same sense. When a baseball pitcher throws a curve, the ball seems to go straight for a while and then "breaks" or "curves."



(a) (b) (c) (d) (e) (f) (g)

Figure 14-6. Curves.

Figure 14-6e is an important type of curve called a broken-line curve. A, B, C, D, E are points of the curve. We say the curve contains or passes through its points. Of course, each curve contains many other points besides those specifically named.

We need a way to distinguish curves such as those shown in Figures 14-6f and 14-6g from other curves. These are called closed curves.

> A closed curve is a plane curve whose representation can be drawn without retracing and with the pencil point stopping at the same point from which it started.
>
> If a closed curve does not intersect itself at any point, we call it a simple closed curve.

Figure 14-6f is a simple closed curve. We could speak of going around the curve and, when we do, we pass through each point just once (except, of course, the starting point).

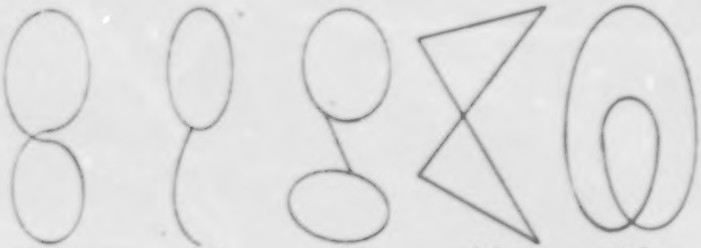

(a) (b) (c) (d) (e)

Figure 14-7. Curves.

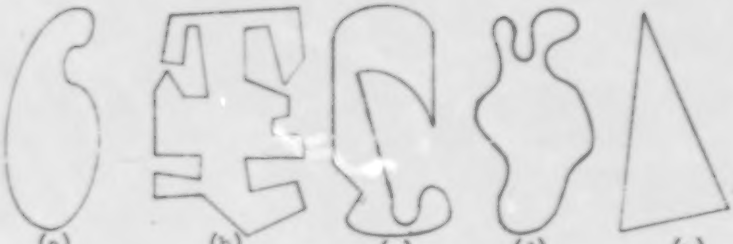

(a) (b) (c) (d) (e)

Figure 14-8. Simple closed curves.

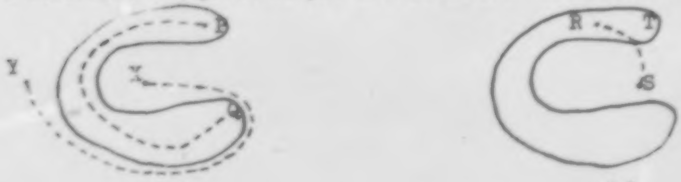
Curves a, d and e in Figure 14-7 are closed curves. Each curve in Figure 14-8 is a simple closed curve, but not one of those in Figure 14-7 is.

From looking at representations of simple closed curves we might guess that any such curve separates the plane into two parts, one of which might be called the interior of the curve and the other the exterior. As a matter of fact this is true and is a very important property of a simple plane closed curve, but one which may not hold if the curve is drawn on another surface. See, for instance, Figure 14-5, where the circle PQR does not separate the surface of the inner tube.

> Every simple plane closed curve separates the plane in which it lies into two parts, the interior of the curve and the exterior of the curve. The curve itself does not belong to either part.

When a line separated a plane into two half-planes we could join any two points in the same half-plane by a segment which did not intersect the line. In like manner, any two points in the interior of the simple closed curve can be joined by a curve which does not intersect it. The same is true for two points in the exterior as illustrated in Figure 14-9a.

Again, if point R is in the interior and point S in the exterior of the curve, we can never join R and S by a plane curve which does not intersect the given simple closed curve. See Figure 14-9b.



(a) (b)

Figure 14-9. Interior and exterior of a simple closed curve.

Sometimes one simple closed curve may be entirely in the interior of a second such curve as in Figure 14-10.

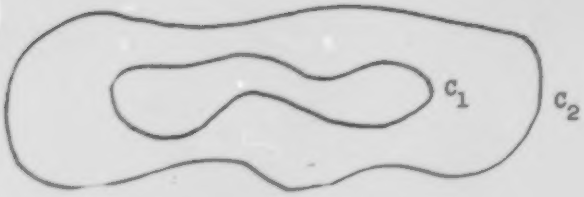


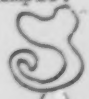
Figure 14-10. C_1 lies in the interior of C_2.

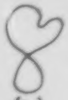
> The interior of any simple closed curve, together with the curve, is called a region. The curve is the boundary of the region.

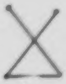
Note that the boundary of the region is part of the region. There are other sets of points which are also called regions. For example, in Figure 14-10 the set of points which are between the two curves, together with the curves, may also be called a region. Its boundary is both curves.

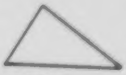
Problems

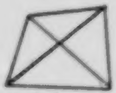
4. Which of the following are closed curves? Which are both closed and simple?

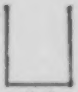

(a)


(b)


(c)


(d)


(e)


(f)

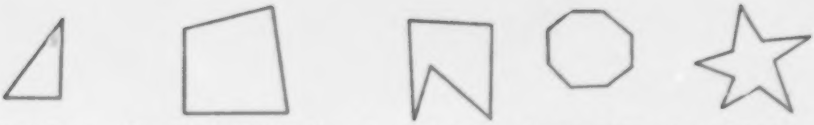
5. Consider the capital letters as ordinarily printed. Which of them are simple closed curves? Which of them separate the plane?

6. Draw a picture of a simple closed curve made up of four segments. Is this a broken-line curve? Do you know a name for such a figure?

Polygons

> If a simple closed curve is the union of three or more line segments, it is called a polygon.

Note that a curve can be the union of three or more segments without being a polygon. See Problem 4 c, e and f. Polygons have special names according to how many line segments are involved. Those with three segments are called triangles, with four, quadrilaterals. After that, the names are made up of the Greek word for the appropriate number followed by the syllable "gon." Thus, pentagon, hexagon, octagon, and decagon are the names for the polygons with 5, 6, 8 or 10 sides respectively.



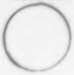
a. triangle b. quadrilateral c. pentagon d. octagon e. decagon

Figure 14-11. Polygons.

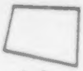
The simplest of the polygons is the triangle. A triangle is the union of three line segments. Let A, B and C be three points, not all on the same line. The triangle ABC, written as $\triangle ABC$, is the union of $\overline{AB}$, $\overline{AC}$ and $\overline{BC}$. Recall that the union of two sets consists of all the elements of the one set together with all the elements of the other. Each of the points A, B and C is called a vertex of the triangle. (The plural of vertex is vertices.) The segments $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$ are called the sides of the triangle. It should be noted that the triangle is the union of three line segments and specifically does not include the interior. We shall study triangles and other polygons again in a later chapter.

Problem

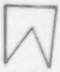
7. Which of the following are polygons?



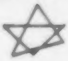
(a)



(b)



(c)



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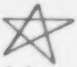

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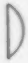

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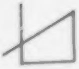

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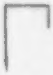
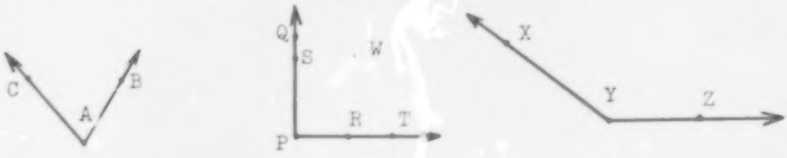


(h)

Definition and Properties of Angles

There is another geometric figure or set of points to be considered now and that is an "angle." An angle is the union of two rays which have the same endpoint but which are not parts of the same line.



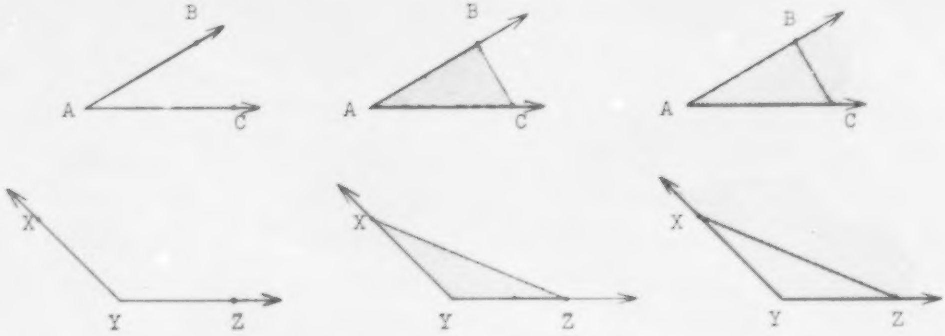
(a) (b) (c)

Figure 14-12. Various angles.

In Figure 14-12a the angle is $\overrightarrow{AC} \cup \overrightarrow{AB}$. Note that in Figure 14-12b the point S is a point of the angle since S is in $\overrightarrow{PQ}$ but point W is not a point of the angle since it is not in either $\overrightarrow{PQ}$ or $\overrightarrow{PT}$.

The common endpoint of the two rays is called the vertex of the angle. The symbol for an angle is $\angle$ and the angle is usually named by naming three points of the angle; the first being a point (not the vertex) on one ray; the second being the vertex; and the third being a point (not the vertex) on the other ray. Thus the angle represented in Figure 14-12a is $\angle BAC$ or $\angle CAB$. In Figure 14-12b the angle may have various names: $\angle QPR$ or $\angle SPT$ or $\angle RPS$, etc. Note that it is correct to say $\angle QPR = \angle TPS$ since they are simply different names for the same angle.

Just as a simple closed curve divides the plane into two parts, the interior and the exterior of the curve, so does an angle divide the plane into two parts, which we shall call the interior and exterior of the angle. But which part shall we call the interior? An easy way to decide is as follows: Consider $\angle BAC$. $\overline{CA}$, $\overline{AB}$ and $\overline{BC}$ determine a simple closed curve, in fact, they determine a triangle which has an interior. $\angle BAC$ divides the plane into two parts. The interior is that one of the two parts which includes the interior of the triangle. Figure 14-13 will help to make this clear.



An $\angle$. Interior of $\triangle$. Interior of $\angle$.

Figure 14-13. Interior of an angle.

Another method of determining the interior of an angle is to use the separation property of a line in a plane. In Figure 14-14, $\angle ABC$ is determined by the rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$. These rays in turn determine the lines $\overleftrightarrow{BA}$ and $\overleftrightarrow{BC}$. $\overleftrightarrow{BA}$ separates the plane into two half-planes and C lies in one of them. We mark this half-plane with horizontal shading. Similarly $\overleftrightarrow{BC}$ separates the plane into two half-planes and A lies in one of them. We mark this half-plane with vertical shading. That portion of the plane which is shaded twice is the interior of $\angle ABC$.

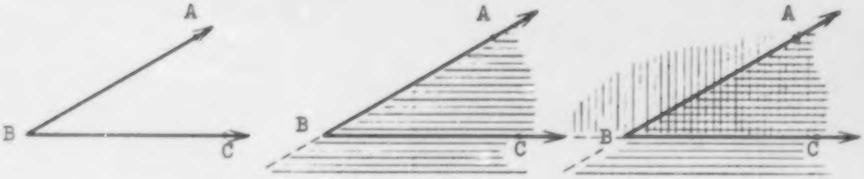


Figure 14-14. Interior of $\angle$ ABC.

If we consider the segments $\overline{AB}$ and $\overline{BC}$ we see that they determine rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$ and th[illegible] determine an angle. It must be remembered that the angle consists of all the points in both rays and not just those points in $\overline{AB}$ and $\overline{BC}$. This is why we should be careful to observe that while a triangle or a polygon determines its angles, the angles are not part of the triangle. As a point set, the $\angle$ ABC consists of the rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$, but the $\triangle$ ABC contains only the points of the segments $\overline{BA}$ and $\overline{BC}$ (as well as those of $\overline{AC}$).

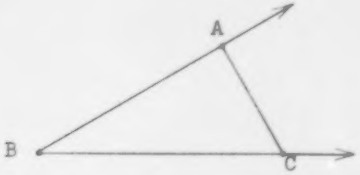
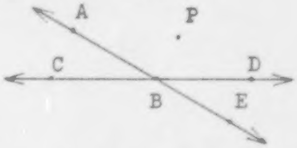


Figure 14-15. $\angle$ ABC is not a subset of $\triangle$ ABC.

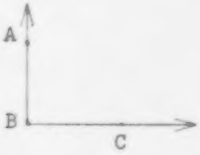
Problems

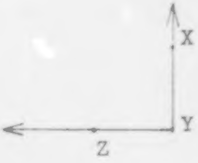


8. In the adjacent figure P lies in the interior of a certain angle. Name the angle and shade its interior.

9. The first definition of the "interior of an angle" was in terms of the "interior of a certain triangle." The second definition did not use the "interior of a triangle." We could therefore define the interior of a triangle" in terms of the "interiors of its angles." Write such a definition.

10. It is incorrect to say $\angle$ ABC = $\angle$ XYZ. Why?

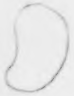
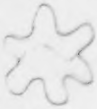
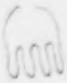




We have been concerned in the last two chapters with various geometric figures or sets of points and some of their properties. These are point, space, curve, line, plane, surface, polygon, angle, etc. We have discussed their relationships, their intersections and separation properties, but we have not mentioned anything about their measures or their sizes. These are, of course, topics of interest and importance and will be taken up in the next chapter.

Exercises - Chapter 14

1. Which of the curves below are simple closed curves?


(a)

(b)

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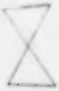
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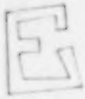
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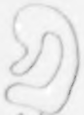
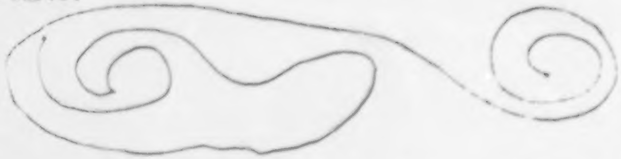
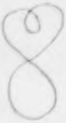
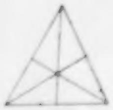

(f)

2. The curve below does not intersect itself. Why is it not a simple closed curve?



3. Which of the curves below are not simple closed curves?


(a)

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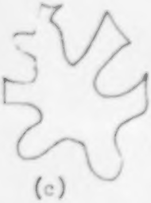
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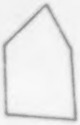
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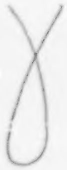
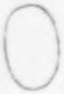
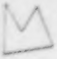
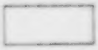

(e)

(f)

4. Draw three examples of each of the following:
   a. curves that are neither closed nor simple.
   b. curves that are closed but not simple.
   c. curves that are both closed and simple.

5. Which of these simple closed curves are pictures of polygons?

(a)


(b)


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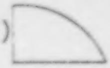

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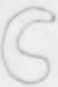

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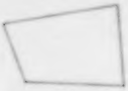

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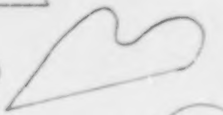

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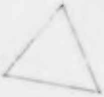

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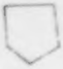
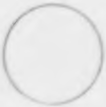
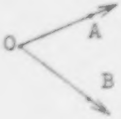
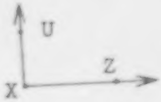


(i)


6. Which figures above are pictures of quadrilaterals?

7. Name the vertex and the sides of each angle below.

(a)


(b)


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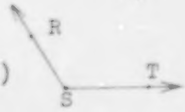

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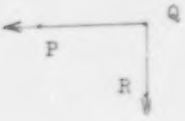
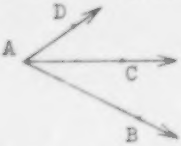
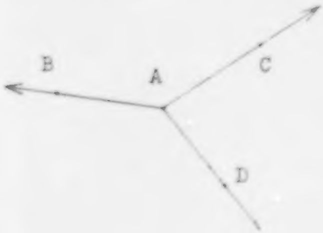


8. Is point C in the interior or exterior of ∠ BAD?

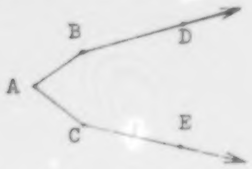


9. Is $\overrightarrow{AC}$ (except for point A) in the interior or in the exterior of ∠ BAD ?

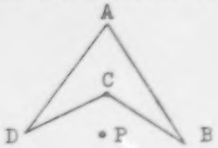
10. Answer the questions of Exercises 8 and 9 in the following case.

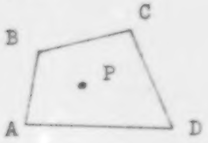


11. Consider the figure below made up of segments $\overline{AB}$ and $\overline{BC}$ and rays $\overrightarrow{BD}$ and $\overrightarrow{CE}$. Does such a figure separate the plane? Does it have an interior? If so, explain how you might decide what the interior is.

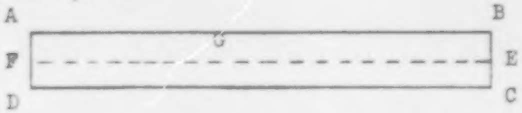


12. In each figure below P is in the interior of both $\angle BAD$ and $\angle BCD$. Is P in the interior of each polygon?



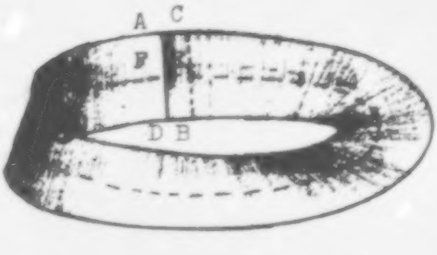


13. Consider a long narrow strip of paper. Draw any closed curve on this paper. Does it separate the strip? Cut along the curve and see if the paper falls apart.



14. Take a similar strip of paper and paste its ends together so that $\overline{BC}$ is matched with $\overline{AD}$ forming a band. Draw a closed curve and cut along it. Will the band always be separated? Try the curve FGE.

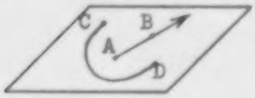
15. Take a similar strip of paper. Paste the ends together after turning $\overline{BC}$ over so that now C falls on A and B on D. Draw a closed curve from E around to E again as indicated by the dotted line in the figure and cut along this curve. Is the band of paper separated? Such a band as this is called a Moebius band. Its curious properties are explored in many semi-popular books on mathematics.



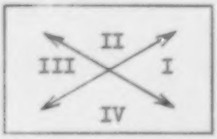
Solutions for Problems

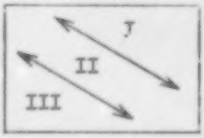
1. There are many sets of planes and lines satisfying the various requirements. We name one set in each case.
    a. ABH and ICD.
    b. ABH and BHD
    c. ABH, BCD and ABC
    d. FHD, GHD and BHD
    e. $\overleftrightarrow{DH}$ and ABC
    f. $\overleftrightarrow{HD}$ and $\overleftrightarrow{BC}$
    g. $\overleftrightarrow{GH}$ and $\overleftrightarrow{BC}$
    h. $\overleftrightarrow{FH}$, $\overleftrightarrow{DH}$ and $\overleftrightarrow{HB}$

2. a. A segment does not separate a plane, nor does a ray. A line does. There is a path from C to D which does not intersect $\overline{AB}$ or $\overrightarrow{AB}$.



    b. Two intersecting lines separate a plane into four parts. Two parallel lines separate a plane into three parts.

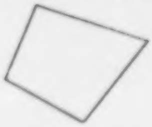


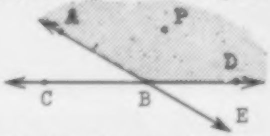


3. a. A half-plane does not separate space nor does a line.
    b. Four

4. (a), (b) and (d) are closed. (a) and (d) are both closed and simple. (c) and (f) are not closed because of the endpoints. (e) cannot be traced by a pencil without retracing some segments or lifting the pencil. Therefore it does not satisfy the definition.

5. D and O are the only simple closed curves. A, B, P, Q and R as well as D and O separate the plane.

6. 
It is a broken-line curve. It is usually called a polygon or a quadrilateral. This figure is considered again in Chapter 15.



7. (b) and (c) are the only polygons.

8. P is in the interior of ∠ ABD.

9. The interior of a triangle is the portion of the interior of each of its angles which is common to all three of them.

10. These are not the same angle and "=" always means two different names for the same object or idea.

Chapter 15

METRIC PROPERTIES OF FIGURES

Introduction

In the last two chapters we studied several geometric concepts. We could not define precisely what we meant by *point*, *line*, *plane* *and* *space* because we found that if we tried it, we eventually ended up in a circle defining *space* in terms of *point* and *point* in terms of *space*. The best we could do was to give descriptions of what we meant, draw pictures to represent the ideas, and discuss properties of and relationships between the various sets of points which were called lines, planes, curves, segments, etc.

We studied how many different points were necessary to determine a line or a plane; how lines and planes might or might not intersect and what their intersections would have to be; how a point, a line and a plane could separate respectively, a line, a plane and space. We looked at curves, particularly at simple closed plane curves which we found had the interesting property of separating the plane into an interior and exterior region. A simple closed curve which is the union of line segments we called a polygon. We found that we can determine a line by specifying two points of the line, and we can determine a plane by specifying three points which were not all in one line. For other figures we may have to specify at least some particular points. For instance: to determine a segment we cannot take any two points in the segment, we must specify the two endpoints; for a ray, we must specify the endpoint and any other point; for an angle, the vertex and then any two other points, one in each ray; and for a triangle we must specify each of the three vertices.

Nowhere in these two chapters have we looked at any properties of geometric figures which needed the idea of size or measure. But now, in order to consider some of the familiar figures such as rectangles and circles, we need to be able to compare two segments so as to say whether the first is longer than the second or the second is longer than the first or whether neither statement is correct. This concept of "longer than" is much the same as the concept "more than" used in comparing two numbers in Chapter 2. At that time, we used the word "equals" and the symbol "=" to mean that we had two names for the same number. Thus,

$8 - 5 = 12 \div 4$ because both $8 - 5$ and $12 \div 4$ are names for the number 3.

In all our work we are going to reserve the word "equals" and the symbol "=" for this idea, that is to say, if A, B, C, etc., are names for points, $A = B$ means that A and B are both names for precisely the same point. $\overline{AB} = \overline{CD}$ means that either $A = C$ and $B = D$ or that $A = D$ and $B = C$ so that we are talking about the same segment.

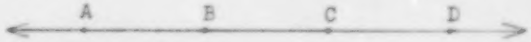


Figure 15-1.

In Figure 15-1 it is not true that $\overline{AB} = \overline{CD}$ since A is not the same point as either C or D.

It is, however, true that $\overleftrightarrow{AB} = \overleftrightarrow{CD}$ because the two different segments do determine the same line. In the same figure, $\overrightarrow{AB} = \overrightarrow{AD}$, since in each case the ray consists of the half-line to the right of A together with the point A.

Congruence of Segments

We come now to the problem: how do we compare two segments $\overline{AB}$ and $\overline{CD}$? Since A and B are fixed locations in space, we cannot move them around. It is possible, however, to have a representation of the line segment on a piece of paper or as a stretched, taut piece of string or as the segments between the tips of a compass. If there is a similar representation for $\overline{CD}$ we can physically compare the two representations. Although we cannot move the segments, we can move their representations. Suppose the tips of a compass are used to represent $\overline{AB}$. Place one tip of the compass on C and see where the other tip falls on the ray $\overrightarrow{CD}$. If it falls between C and D, we say that $\overline{CD}$ is longer than $\overline{AB}$. If the tip of the compass falls on D, $\overline{AB}$ is congruent to $\overline{CD}$. If it falls beyond D, $\overline{AB}$ is longer than $\overline{CD}$.

It is perhaps a bit difficult to see why we want to say that segments are "congruent" instead of using the more familiar word "equal." But strictly speaking the familiar usage "two lines are equal" is inaccurate on two counts. First of all, it is clearly "line segments" rather than "lines" that are being referred to. Secondly, and more to the point here, what is really being said is that the two segments have the same "length" or "measurement." The "length of a line segment" is a number which measures the segment in a sense which will be discussed in Chapter 16, and to say that two lengths are equal is to say that the same number is the measure of each. But, without this idea at all, it makes perfectly good sense to say that $\overline{AB}$ is congruent to $\overline{CD}$, with the meaning attached to this statement in the previous paragraph, and this is all we need at the moment.

Notice that this method of comparing two segments is not the same as, but bears a resemblance to the method of comparing two sets in Chapter 2. If we tried to match the elements of set P with those of set Q and ran out of elements of P before we did those of Q so that we matched the elements of P with those of a proper subset of Q, we said Q was "more than" P. It was also possible for P to be more than Q or for P to "match" Q. These relationships between two sets roughly correspond to the relationships between two segments when they are compared as we did a moment ago.

For brevity in writing, symbols are needed to represent these relationships between segments. Since the symbol "=" has been reserved to indicate two different names for the same thing, whether it be a number, a set, a segment, a concrete object or an idea, we need a new symbol for congruence. To indicate that $\overline{AB}$ and $\overline{CD}$ are congruent we write $\overline{AB} \cong \overline{CD}$. If $\overline{CD}$ is longer than $\overline{AB}$, we see that $\overline{AB}$ is

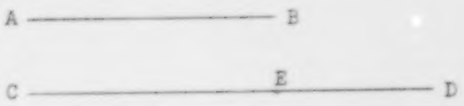


Figure 15-2. $\overline{CD} > \overline{AB}$.

congruent to a subset of $\overline{CD}$, $\overline{CE}$ in Figure 15-2. We write this $\overline{CD} > \overline{AB}$ (or $\overline{AB} < \overline{CD}$). Note that this use of the symbols ">" and "<" indicates a comparison relationship between line segments. In Chapter 2 we used the same symbols to indicate a comparison relationship between two numbers. We should keep carefully in mind the difference between the two usages. Usually it will be easy to see when we are talking about segments and when we are talking about numbers.

When we compare two segments $\overline{AB}$ and $\overline{CD}$ by the method outlined above, only three things can happen: the tip of the compass, representing B, can fall between C and D, or on D or beyond D. Accordingly we find that one and only one of the three statements $\overline{AB} < \overline{CD}$, $\overline{AB} \cong \overline{CD}$

and $\overline{AB} > \overline{CD}$ will be true.

If we compare $\overline{AB}$ with two other segments $\overline{CD}$ and $\overline{EF}$, as shown in Figure 15-3, we may find that $\overline{CD} > \overline{AB}$ and $\overline{AB} > \overline{EF}$. If we now compare $\overline{CD}$ and $\overline{EF}$, we will always find that $\overline{CD} > \overline{EF}$. Thus

If $\overline{CD} > \overline{AB}$, and $\overline{AB} > \overline{EF}$, then $\overline{CD} > \overline{EF}$.

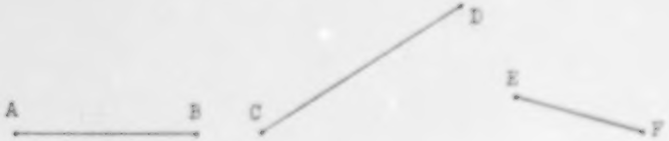


Figure 15-3. Comparing three segments.

Suppose we have a segment $\overline{CD}$ and a point A fixed on a line $\overleftrightarrow{AP}$, as in Figure 15-4. Using the tips of a compass to represent $\overline{CD}$, we can place one tip on A and with the other tip determine two points of $\overleftrightarrow{AP}$, say R and S, one on each side of A so that $\overline{AR} \cong \overline{CD}$ and $\overline{AS} \cong \overline{CD}$. Thus we can determine two segments with endpoint A on line $\overleftrightarrow{AP}$ congruent to $\overline{CD}$.

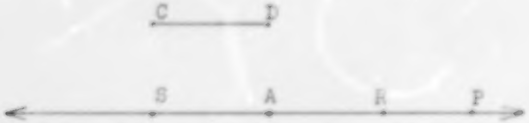


Figure 15-4. $\overline{AS} \cong \overline{CD}$; $\overline{AR} \cong \overline{CD}$.

If, as in the above figure, $\overline{SA} \cong \overline{AR}$ and A is between S and R, we say that point A bisects $\overline{SR}$.

Problems*

1. If $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$, is it true that $\overline{AB} \cong \overline{EF}$? Why or why not? Is $\overline{CD} \cong \overline{AB}$?
2. If $\overline{AB} > \overline{CD}$ and $\overline{AB} > \overline{EF}$ is it true that $\overline{CD} > \overline{EF}$? Why or why not? Is $\overline{CD} > \overline{AB}$?

---

* Solutions for problems in this chapter are on page 183.

Congruence of Angles

We can compare two angles in much the same way as we compared two segments. Thus if two angles $\angle ABC$ and $\angle PQR$ are given, we can take as a representation of $\angle ABC$ a tracing, say $\angle A'B'C'$, and make the ray $\overrightarrow{B'C'}$ fall on $\overrightarrow{QR}$ with $\overrightarrow{B'A'}$ falling in the same half-plane as $\overrightarrow{QP}$ and with B' falling on Q. See Figure 15-5. Now if $\overrightarrow{B'A'}$ falls on $\overrightarrow{QP}$ we say that $\angle ABC \cong \angle PQR$.

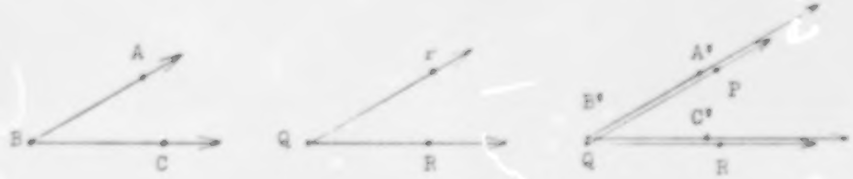


Figure 15-5. $\angle ABC \cong \angle PQR$.

But $\overrightarrow{B'A'}$ may fall in the interior of $\angle PQR$ in which case $\angle ABC$ is less than $\angle PQR$ or $\angle ABC < \angle PQR$. See Figure 15-6.

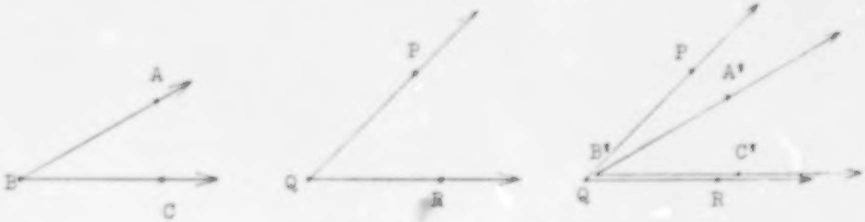


Figure 15-6. $\angle ABC < \angle PQR$.

Finally $\overrightarrow{A'B'}$ may fall in the exterior of $\angle PQR$ and in this case $\angle ABC$ is greater than $\angle PQR$ or $\angle ABC > \angle PQR$. See Figure 15-7.

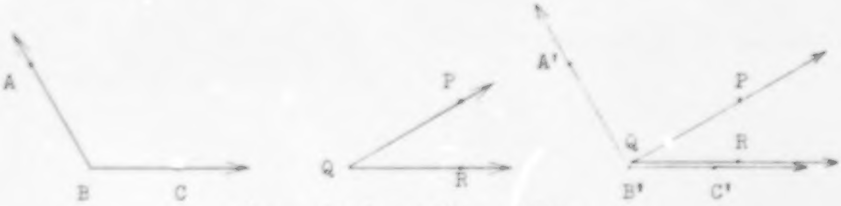
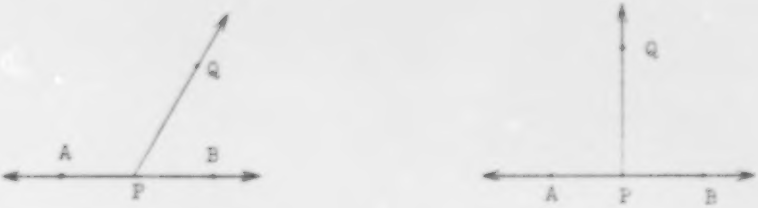


Figure 15-7. $\angle ABC > \angle PQR$.

Again as in comparing segments we find that these three are the only possibilities. That is: either $\angle ABC < \angle PQR$ or $\angle ABC \cong \angle PQR$ or $\angle ABC > \angle PQR$.

There is a special angle of great importance called a right angle which we may illustrate as follows. Take any point P on $\overleftrightarrow{AB}$ such that A and B are on opposite sides of P and take any point Q not on $\overleftrightarrow{AB}$. Draw $\overrightarrow{PQ}$. This is illustrated in Figure 15-8. We get two angles $\angle APQ$ and $\angle BPQ$. If we compare these angles we may occasionally find that they are congruent. In this case we say that $\angle APQ$ and $\angle BPQ$ are both right angles.



$\angle APQ > \angle BPQ$

$\angle APQ \cong \angle BPQ$

$\angle$'s APQ and BPQ are right angles.

Figure 15-8.

Models of right angles occur in many places in the world as, for instance, at the corner of an ordinary sheet of paper. A model can be made even from an irregular sheet of paper by folding it twice, as shown in Figure 15-9.

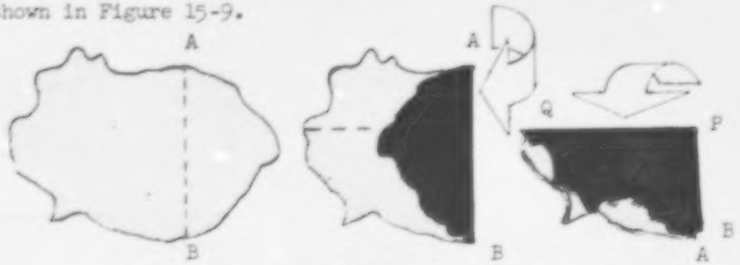
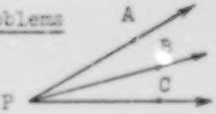


Figure 15-9. Folding an irregular shape to get a model of a right angle.

Problems

3. 

If $\angle APB \cong \angle BPC$, are these angles right angles? Why or why not?

4. If $\angle ABC > \angle PQR$ and $\angle PQR > \angle XYZ$, what can be said about $\angle XYZ$ and $\angle ABC$?

5. What is wrong with each of the following:
$\overline{AB} \cong \overrightarrow{CD}$; $\overleftrightarrow{AB} = \overrightarrow{AB}$; $\angle ABC > \overline{BC}$?

Convex Polygons

Now that we can compare two segments and decide whether they are congruent or not and also compare two angles, we are able to learn a little more about polygons.

If we consider quadrilaterals and other polygons with more than three sides, we soon see that two quite different situations are possible. These are illustrated in Figure 15-10 where the polygons represented in a, b, d surely have a different character from those in c and e.

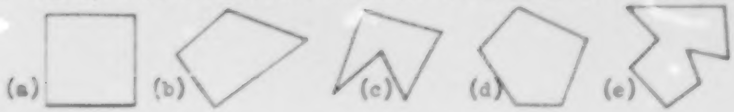


Figure 15-10. Various polygons.

How shall we distinguish these? Perhaps the easiest way is to observe that in a, b and d the interior of each polygon belongs to the interior of each angle determined by a vertex of the polygon and the two segments of which that vertex is an endpoint. This is not the case in c and e.

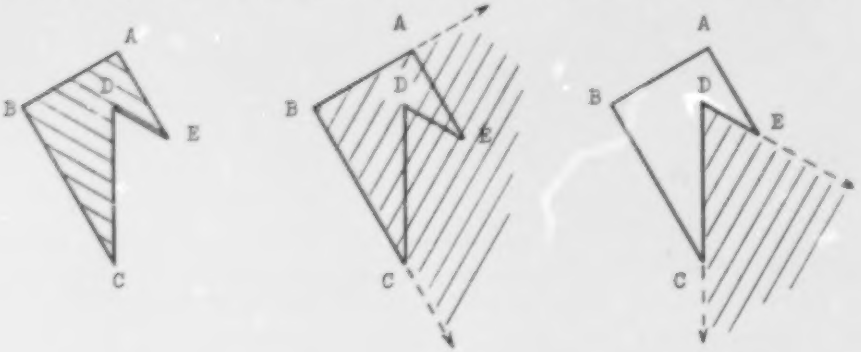


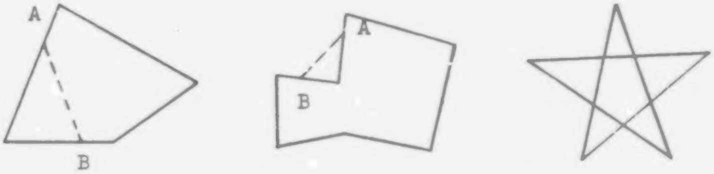
Figure 15-11. The interior of a polygon.

> If the interior of a polygon is part of the interior of each angle determined by a vertex and the two segments of which that vertex is an endpoint, then that polygon is called convex.

Another way to define a convex polygon is as follows. See Figure 15-12.

> A polygon is convex if for any two points A and B on the polygon, $\overline{AB}$ contains besides A and B only points in the interior of the polygon.

Usually, it is convex polygons that we are interested in.



Simple convex polygon.

Simple non-convex polygon.

Neither simple nor convex. Not a polygon.

Figure 15-12.

Problem

6. Is it true that all triangles are convex?

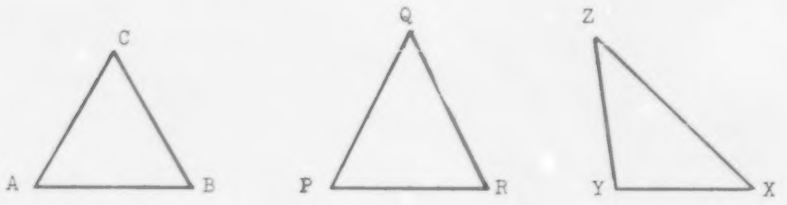
Classification of Triangles

Triangles can be classified by comparing their sides or their angles. (We really should not speak of an angle of a triangle, since we saw above that the set of points which is the angle is not a subset of the set of points which is a triangle. But since the segments of the triangle do determine three angles, we use the words "angle of a triangle" for the more correct but much longer "angle whose rays are determined by the sides of a triangle.")

First:

Considering the sides of a triangle.

a. If all three sides are congruent, the triangle is equilateral.
b. If two sides are congruent, the triangle is isosceles.
c. If no two sides are congruent, the triangle is scalene.



Equilateral $\overline{AB} \cong \overline{BC} \cong \overline{AC}$

Isosceles $\overline{PQ} \cong \overline{RQ}$

Scalene $\overline{XY} < \overline{YZ} < \overline{XZ}$

Figure 15-13. Triangles classified according to sides.

Second:

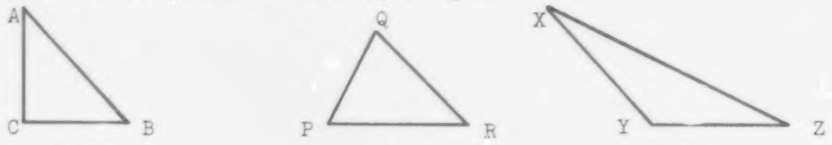
Considering the angles of a triangle,

a. One angle may be a right angle. Such a triangle is called a *right triangle*.

b. All angles may be less than a right angle. In this case the triangle is said to be *acute*.

c. One angle may be greater than a right angle. Such a triangle is called an *obtuse triangle*.

In cases (a) and (c) comparison of angles will show that the other two angles are always less than a right angle.



Right $\angle ACB$ is right

Acute Each angle < right angle

Obtuse $\angle XYZ >$ right angle

Figure 15-14. Triangles classified by angles.

It is interesting to note that if you compare the angles of an equilateral triangle you will find that they are all congruent and if you do so for the angles of an isosceles triangle you will find that two of them are congruent. Can you determine which two? Is it also true that if two or three angles of a triangle are congruent then two or three sides will be? If you try this with some models, you will find that it seems to be true. As a matter of fact it really is true.

Problems

7. Sketch: a. an obtuse triangle; b. a triangle which is both obtuse and isosceles; c. an acute, scalene triangle.
8. An equilateral triangle has been defined above. Now define an equiangular triangle.

Classification of Quadrilaterals

Triangles are always convex, but other polygons may not be. However, usually it is convex polygons that we are interested in. There are many interesting types of convex polygons. Those with four sides, the quadrilaterals, may be classified as were triangles by listing special properties of their sides or angles. Such a classification is given below for a few of the best known cases.

(a) The scalene quadrilateral. None of its sides are congruent.

(b) The parallelogram. Its opposite (non-intersecting) sides are segments of parallel lines. They are also always congruent.

(c) The rectangle. It is a parallelogram whose angles are all congruent. They are also all right angles.

(d) The square. It is a rectangle whose sides are all congruent.

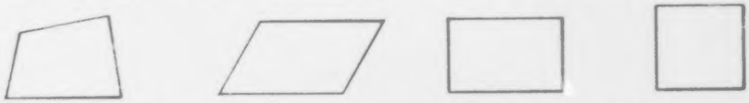

(a) scalene (b) parallelogram (c) rectangle (d) square

Figure 15-15. Special quadrilaterals.

Circles

The properties and relationships of triangles, quadrilaterals and polygons with a greater number of sides are studied at length in plane geometry. But now let us shift our attention to another simple closed curve with which you are probably quite familiar. This is the circle. What is a circle? You know that to draw a representation of a circle you put the metal tip of a compass at the point you want for the center and keeping it fixed and the spread of the compass unchanged, draw the curve. The segments from the center to all the points of the curve are congruent. Thus points A and B belong to the same circle with center O if and only if $\overline{OA} \cong \overline{OB}$. More formally:

> A circle is a simple closed curve having a point O in its interior and such that if A and B are any two points in the curve $\overline{OA} \cong \overline{OB}$.

The center is not a point of the circle. In Figure 15-16 point O is the center of the circle.

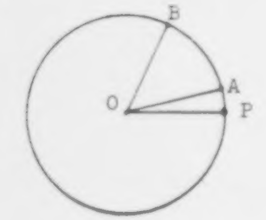


Figure 15-16. Circle.

> A line segment with one endpoint at the center of the circle and the other endpoint on the circle is called a radius.

$\overline{OP}$, $\overline{OA}$ and $\overline{OB}$ are all radii. Clearly all radii (radii is the plural of radius) of a given circle are congruent. $\overline{OP} \cong \overline{OA} \cong \overline{OB}$.

Sometimes we name a simple closed curve by a letter, as "the circle C." When we say "the circle C," C is not necessarily a point of the circle. In fact frequently the circle is named by its center point.

A circle is a simple closed curve. Consequently, it has an interior and an exterior. Suppose we have a circle with the center at the point P and with radius $\overline{PR}$. A point, such as A, is inside the circle, if $\overline{PA} < \overline{PR}$, while a point, such as B, is outside the circle if $\overline{PB} > \overline{PR}$. Thus it is easy, in the case of the circle, to describe precisely what its interior and its exterior are. The interior is the set of all points A such that $\overline{PA} < \overline{PR}$. The exterior is the set of all points B such that $\overline{PB} > \overline{PR}$.

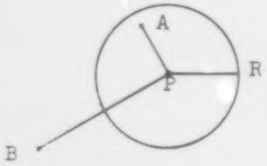


Figure 15-17.

Now, shade lightly the interior of the circle as shown. What is the union of the circle and its interior? The union of a simple closed curve and its interior is called a "region." The union of the circle and its interior is a

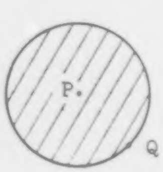


Figure 15-18.

circular region. Note that the circle is the curve only, while the circular region includes the curve and its interior.

The word diameter is closely associated with the word radius.

> A *diameter* of a circle is a line segment which contains the center of the circle and whose endpoints lie on the circle.

For the circle represented by the figure at the right, three diameters are shown: $\overline{AB}$, $\overline{MN}$ and $\overline{VW}$. (A diameter of a circle is the longest line segment that can be drawn in the interior of a circle such that its endpoints are on the circle.) How many radii are shown in the figure?

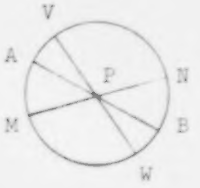


Figure 15-19.

In Chapter 14 we saw that a point in a line separated the line into two half-lines. Consider a similar question with respect to a circle. Does point Q separate the circle in Figure 15-20 into two parts? If we start at Q and move in a clockwise direction we will, in due time, return to Q. The same is true if we move in a counter-clockwise direction. A single point does *not* separate a circle into two parts.

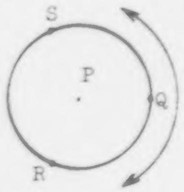


Figure 15-20.

In Figure 15-21, the two points, X and Y, do separate the circle into two parts called *arcs*. One of the arcs contains the point A. The other part contains B. No path from X to Y along the circle can avoid at least one of the points A and B. Thus, we see it takes two different points to separate a circle into two distinct parts. The arcs are written $\widehat{XAY}$ and $\widehat{XBY}$ or sometimes just $\widehat{XY}$ if it is clear from the context as to which arc is meant.

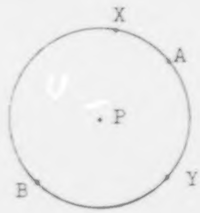


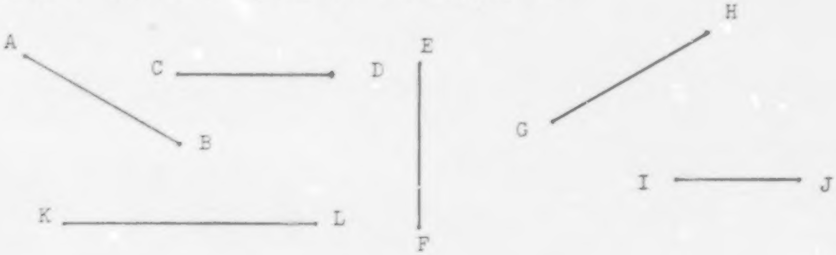
Figure 15-21.

Summary

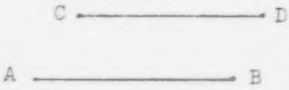
We have considered in this chapter methods of comparing segments and angles. This has enabled us to identify and classify some familiar geometric figures such as triangles, quadrilaterals and circles. Remember that there is, as yet, no way to measure segments or angles, that is to say how "long" a segment is or how "big" an angle is. To "measure" a segment or an angle is to assign a number to it in some way. The important problem of how this may be done is the subject of the next chapter.

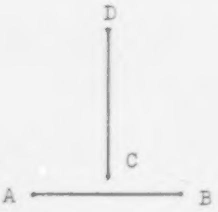
Exercises - Chapter 15

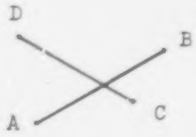
1. Which of the following segments are congruent?



2. In each of the following pairs of segments determine if $\overline{AB} < \overline{CD}$, $\overline{AB} > \overline{CD}$ or $\overline{AB} \cong \overline{CD}$.

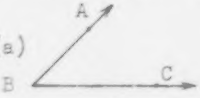




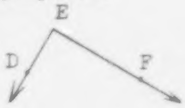


3. Which angles are right angles? Do not guess but make a model of a right angle for comparison purposes.

(a)



(b)



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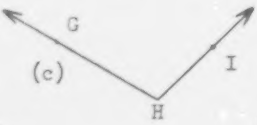

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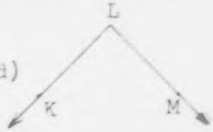

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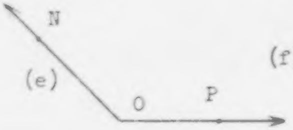

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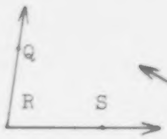
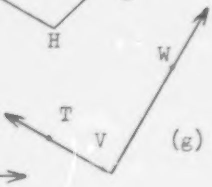
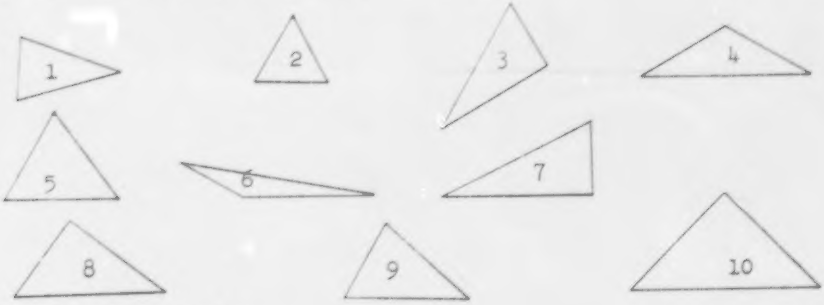


(g)

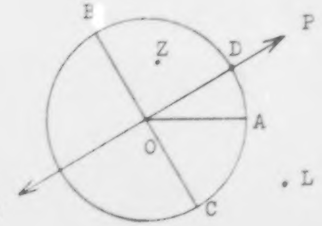


4. In the figures for Exercise 3, (a) compare ∠ DEF with each of the others. (b) compare ∠ GHI with each of the others.

5. (a) Classify the following triangles by considering their sides.
   (b) Classify them by considering their angles.



6. Given the circle below:



   a. Name two radii of the circle.
   b. Name the diameter of the circle.
   c. Is point Z in the interior or exterior region?
   d. Is point L in the circular region?
   e. Name four points in the circular region.
   f. Are $\overline{OA}$ and $\overline{OC}$ congruent?
   g. Name three different arcs with endpoint A.

7. a. Does a straight line separate a plane?
   b. Does a circle separate a plane?
   c. Does a point separate a straight line?
   d. Does a point separate a circle?

8. a. What is a simple closed polygon?
   b. What is a convex polygon?
   c. What is an equilateral triangle?
   d. What is a circle?

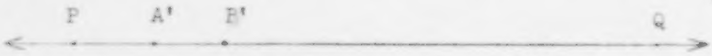
9. O is the center of a circle and $\overline{OA}$ is a radius. If $\overline{OA} > \overline{OB}$ does B lie in the circle or in the interior of the circle?

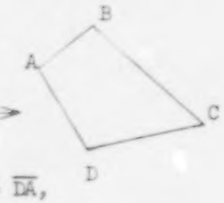
10. Two circles are given both with center O. $\overline{OA}$ is a radius of one and $\overline{OB}$ a radius of the other. If $\overline{OA} < \overline{OB}$ what is true about the circles?

11. Fold a piece of paper twice to get a model of a right angle. How many right angles can you fit together side by side around a point in the plane?

12. If two circles with centers A and B are given, two rather different methods could be thought of to describe congruence of circles: (a) in terms of matching representations of the circles, (b) in terms of comparing the radius of one with that of the other. Discuss the two methods.

13. Consider polygon ABCD. Take a point A' on line $\overleftrightarrow{PQ}$.





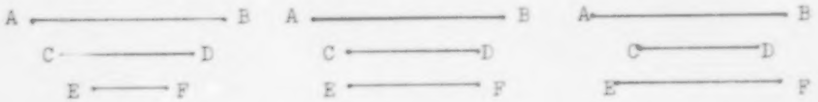
Make $\overline{A'B'} \cong \overline{AB}$ and $\overline{B'C'} \cong \overline{BC}$, $\overline{C'D'} \cong \overline{CD}$ and $\overline{D'E'} \cong \overline{DA}$, being sure that the segments intersect only at their endpoints. What is the union of these segments?

14. For any one circle, what is the intersection of all its diameters?

Solutions for Problems

1. If $\overline{AB} \cong \overline{CD}$, then a representation of $\overline{AB}$ can be made to coincide with $\overline{CD}$. The representation of $\overline{AB}$ will then also serve as a representation of $\overline{CD}$ and since $\overline{CD} \cong \overline{EF}$, it can be made to coincide with $\overline{EF}$. This shows that $\overline{AB} \cong \overline{EF}$. Yes, $\overline{CD} \cong \overline{AB}$.

2. No, because $\overline{CD}$ is congruent to a part of $\overline{AB}$ and $\overline{EF}$ is also congruent to a part of $\overline{AB}$, but we do not know whether then two parts are congruent or not. It may be that $\overline{CD} < \overline{EF}$ or $\cong \overline{EF}$ or $> \overline{EF}$.



No, only one of $\overline{AB} > \overline{CD}$, $\overline{AB} \cong \overline{CD}$, $\overline{AB} < \overline{CD}$ can be true and we already know that $\overline{AB} > \overline{CD}$.

3. The angles are not right angles. For right angles it is necessary that A, P and C lie on one straight line. See the definition.

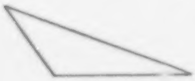
4. $\angle ABC > \angle XYZ$.

5.

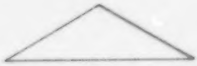
$\overline{AB} \cong \overrightarrow{CD}$.	A segment cannot be congruent to a ray though $\overline{AB}$ might be $\cong \overline{CD}$.
$\overleftrightarrow{AB} = \overrightarrow{AB}$.	"=" means different name for the same object and the line $\overleftrightarrow{AB}$ is not the same object as the ray $\overrightarrow{AB}$.
$\angle ABC > \overline{BC}$.	Comparisons are made between objects of the same kind. Two angles may be compared or two segments but not an angle and a segment.

6. Yes, all triangles are convex. This is hard to prove but easy to see by drawing a few figures.

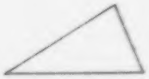
7.



obtuse



obtuse, isosceles



acute, scalene

8. An equiangular triangle is a triangle whose three angles are congruent to each other.

Chapter 16

LINEAR AND ANGULAR MEASURE

Introduction

In the first twelve chapters of this book we were concerned with the idea of number. We talked about whole numbers, how to use them in counting the members of a set, how to add, subtract, multiply and divide them. In the last three chapters we studied geometric figures but without measuring them. We come now to the problem of measurement. What do we mean by measurement?

To measure a line segment is to assign a number to it. This cannot be done by counting the points of the segment since there are infinitely many points in any segment. We have to develop some new concept to take the place of counting. We want to compare the "sizes" of two segments and this is done by comparing each segment with a certain arbitrary unit in a manner which is the subject of this chapter. When this concept of "measurement" has been developed we will find that it is applicable not only to line segments but in a closely related fashion to angles, areas of regions, volume of solids, time, work, energy and many other concepts or physical objects.

Measurement is really one of the connecting links between the physical world around us and mathematics. So is counting, but in a different way. We count the number of books on the desk, but measure the length of the desk. We count the number of hot dogs we need for a picnic, but measure the amount of milk we need.

The Measure of a Segment

Let us start by considering how to measure a segment. In Chapter 15 we saw how to compare two line segments $\overline{AB}$ and $\overline{CD}$ in order to say whether $\overline{AB} > \overline{CD}$ or $\overline{AB} \cong \overline{CD}$ or $\overline{AB} < \overline{CD}$.

When $\overline{AB}$ and $\overline{CD}$ can be conveniently represented by a drawing on a piece of paper, this comparison can be carried out, at least approximately, by tracing a copy of $\overline{AB}$ and placing it on top of the drawing of $\overline{CD}$. But, even if $\overline{AB}$ and $\overline{CD}$ were much too long or much too microscopically short to be drawn satisfactorily on a sheet of paper at all, it would still be possible to conceive of $\overline{AB}$ and $\overline{CD}$ as being such that exactly one of the following three statements is true:

1. $\overline{AB} < \overline{CD}$
2. $\overline{AB} \cong \overline{CD}$
3. $\overline{AB} > \overline{CD}$

In mathematics, we think of the endpoints A and B of any given line segment as being exact locations in space, although these endpoints can be represented only approximately by penciled dots. Similarly, $\overline{AB}$ is considered as having a certain exact length, although this length can be determined only approximately, by "measuring" a drawing representing $\overline{AB}$.

Let us describe the process of measurement. The first step is to choose a line segment, say $\overline{RS}$, to serve as unit. This means to select $\overline{RS}$ and agree to consider its measure to be exactly the number 1.

We should recognize that this selection of a unit is an arbitrary choice we make. Different people might well choose different units and historically they have, giving rise to much confusion. For example, at one time the English "foot" was actually the length of the foot of the reigning king and the "yard" the distance from his nose to the end of his outstretched arm. Imagine the confusion when the king died if the next one was of much different stature. Various standard units will be discussed a little later but meanwhile we return to the choice of $\overline{RS}$ as our unit, recognizing the arbitrariness of this choice.

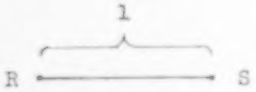


Figure 16-1. The unit $\overline{RS}$

Now it is possible to conceive of a line segment, $\overline{CD}$, such that the unit $\overline{RS}$ can be laid off exactly twice along $\overline{CD}$, as suggested in Figure 16-2.

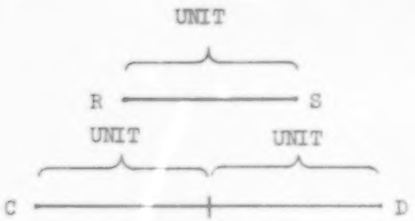


Figure 16-2.

Then by agreement the measure of $\overline{CD}$ is the number 2 and the length of $\overline{CD}$ is exactly 2 units although $\overline{CD}$ can be represented only approximately by a drawing. In the same way, line segments of length exactly 3 units, or exactly 4 units, or exactly any larger number of units are conceptually possible, although such line segments can be drawn only approximately. In fact, if a line segment is very long--say a million inches long--no one would want to try to draw it even approximately; but such a segment can still be thought of.

We can also conceive of a line segment, $\overline{AB}$, such that the unit $\overline{RS}$ will not "fit into" $\overline{AB}$ a whole number of times at all. In Figure 16-3 $\overline{AB}$ is a line segment such that starting at A the unit $\overline{RS}$ can be laid off 3 times along $\overline{AB}$ reaching Q which is between A and B, although if it were laid off 4 times we would arrive at a point P which is well beyond B. What can be said about the length of $\overline{AB}$? Well, surely $\overline{AB}$ has length *greater than* 3 units and *less than* 4 units. In this particular case, we can also estimate visually that the length of $\overline{AB}$ is nearer to 3 units than to 4 units, so that *to the nearest unit* the length of $\overline{AB}$ is 3 units. This is the best we can do without considering fractional parts of units, or else shifting to a smaller unit.

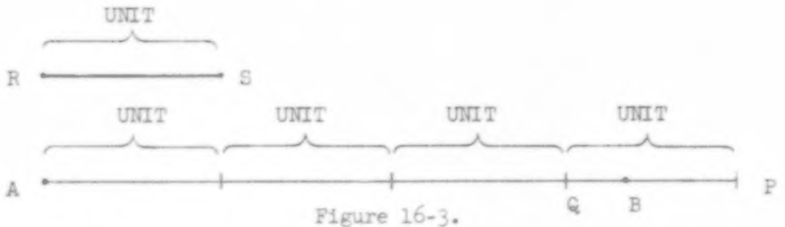
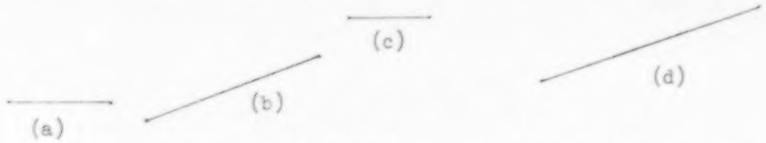


Figure 16-3.

Problems*

1. Using the unit ——— find the measure of each of the following segments to the nearest unit.



2. Using the unit ——— find the measure of each of the segments in Problem 1 to the nearest unit.

To help us in estimating whether the measure of a segment is say, 3 or 4, we need to bisect our unit. In Figure 16-4 $\overline{RS}$ is again shown as unit with T bisecting $\overline{RS}$ so that $\overline{RT} \cong \overline{TS}$ and $\overline{RS}$ is used to measure $\overline{MN}$.

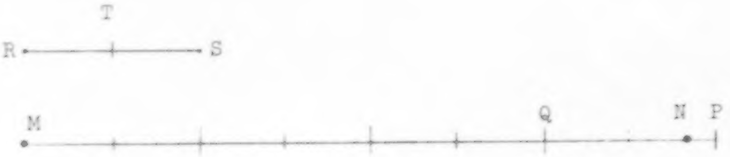


Figure 16-4.

In laying off the unit along $\overline{MN}$, label P the endpoint of the first unit that falls on or beyond N and Q the end of the preceding unit just as you did for $\overline{AB}$ in Figure 16-3. Using $\overline{RT}$ (which has just been determined) in Figure 16-3, we can check that $\overline{BP} > \overline{RT}$ and then the measure of $\overline{AB}$ is 3. In Figure 16-4, $\overline{NP} < \overline{RT}$ and the measure of $\overline{MN}$ is 4. There is nearly always a decision to be made about whether or not to count the last unit which extends beyond the endpoint of the segment being measured. The reason for this is that it is rare indeed for the unit to fit an exact number of times from endpoint to endpoint. It is well to realize now that measurement is approximate and subject to error. The "error" is the segment from the end of $\overline{AB}$ to the end of the last unit being counted. In Figure 16-3 the error is $\overline{BQ}$, in

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* Solutions for problems in the chapter are on page 200.

Figure 16-4, it is $\overline{NP}$. We note that the error in any measurement is always at most half the unit being used.

Let us emphasize one thing about terminology. In a phrase similar to "a line segment of length 3 units" we mean "the measure of the line segment in terms of a particular unit is the number 3." The point here is simply to have a way of referring to the numbers involved so that they can be added, multiplied, etc. Remember that we have learned how to apply arithmetic operations only to numbers. You don't add yards any more than you add apples. If you have 3 apples and 2 apples, you have 5 apples altogether, because

$$3 + 2 = 5.$$

You add numbers, not yards or apples.

As we shall see shortly, the use of different units gives rise to different measures for the same segment. Thus, if we consider in Figure 16-5 a segment congruent to $\overline{MN}$ in Figure 16-4 but use $\overline{KL}$ as our unit, $\overline{MN}$ has a length of 6 units.

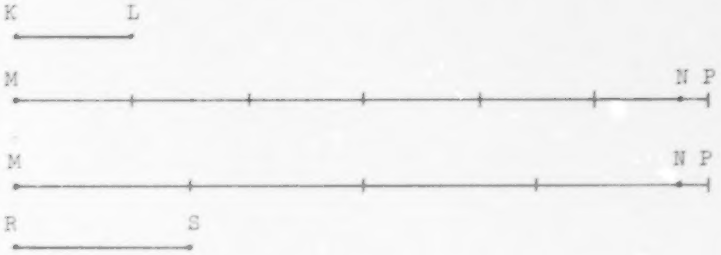


Figure 16-5.

Standard Units

Numbers of people each using their own units would have difficulty comparing their results or communicating with each other. For these reasons certain units have been agreed upon by large numbers of people and such units are called standard units.

Historically there have been many standard units, such as a yard, an inch or a mile used to measure line segments. Such a variety is a great convenience. An inch is a suitable standard unit for measuring the edge of a sheet of paper, but hardly satisfactory for finding the length of the school corridor. While a yard is a satisfactory standard for measuring the school corridor, it would not be a sensible unit for finding the distance between Chicago and Philadelphia.

Such units of linear measure as inch, foot, yard and mile are standard units in the British-American system of measures. In the eighteenth century in France, a group of scientists developed the system of measures which is known as the metric system using a new standard unit.

In the metric system, the basic standard unit of length is the *meter*, which is approximately 39.4 inches or a little more than 1 yard. Thus, in the Olympic Games, where the metric system is used, we have the 100 meter dash, which is just a little longer than the 100 yard dash. The metric system is in common use in all countries except those in which English is the main language spoken and is used by all scientists in the world including those in English speaking countries.

The principal advantage of the metric system over the British-American system lies in the fact that the metric system has been designed for ease of conversion between the various metric units by exploiting the decimal system of numeration. Instead of having 12 inches to the foot, 3 feet to the yard and 1760 yards to the mile, the metric system has 10 centimeters to a decimeter, 10 decimeters to a meter and 1,000 meters to a kilometer. This makes conversions between units very easy.

We have already noted that in the metric system, the *meter* is the unit which corresponds approximately to the yard in the British-American system. The metric unit which corresponds to the inch is the centimeter which is one-hundredth of a meter. A meter is almost 40 inches so it takes about $2\frac{1}{2}$ centimeters to make an inch or to put it another way a centimeter is about $\frac{2}{5}$ or .4 of an inch. Figure 16-6 illustrates a scale of inches and a scale of centimeters so you can compare them.

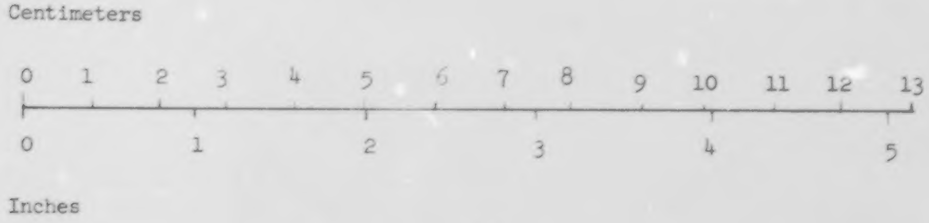


Figure 16-6.

So far we have said nothing about metric units larger than the meter. The most useful of these is the *kilometer*, which is defined to be 1,000 meters. The kilometer is the metric unit which closely corresponds to the British-American mile. It turns out that one kilometer is a little more than six-tenths of a mile. The Olympic race of 1500 kilometers is over a course which is just a little less than a mile.

We have treated the inch, foot, yard and mile as "standard" units for linear measure, in contrast to units of arbitrary size, which may be used when communication is not important. Actually, the one standard unit for linear measure even in the United States is the meter, and the correct sizes of other units such as the centimeter, inch, foot and yard are specified by law with reference to the meter. Various methods for maintaining a model of the standard meter have been used by the Bureau of Standards. For many years the model was a platinum bar, kept under carefully controlled atmospheric conditions. The meter is now defined as having length which is 1,650,763.73 times the wave length of orange light from krypton 86. This standard for the meter is preferred because it can be reproduced in any good scientific laboratory and should provide a more precise model than the platinum bar.

Scales and Rulers

Once a standard unit such as a yard, meter or mile is agreed upon, the creation of a scale greatly simplifies measurement.

> A scale is a number line with the segment from 0 to 1 congruent to the unit being used.

A scale can be made with a non-standard unit or with a standard unit.

> A ruler is a straight edge on which a scale using a standard unit has been marked.

If we use the inch as the unit in making a ruler, we have a measuring device designed to give us readings to the nearest inch. Most ordinary rulers are marked with unit one-sixteenth of an inch or with unit one millimeter.

The Approximate Nature of Measure

Any measurement of the length of a segment made with a ruler is, at best, approximate. When a segment is to be measured, a scale based on a unit appropriate to the purpose of the measurement is selected. The unit is the segment with endpoints at two consecutive scale divisions of the ruler. The scale is placed on the segment with the zero-point of the scale on one endpoint of the segment. The number which corresponds to the division point of the scale nearest the other endpoint of the segment is the measure of the segment. Thus, *every measurement is made to the nearest unit*. If the inch is the unit of measure for our ruler, then we have a situation in which two line segments, apparently not the same length, may have the same measure, to the nearest inch.

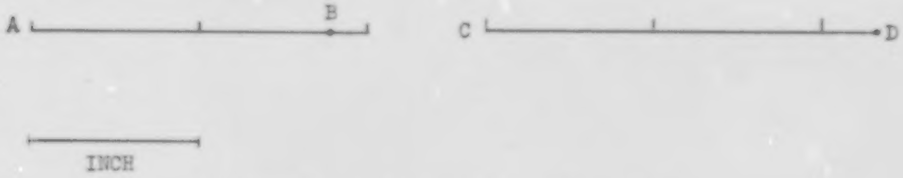


Figure 16-7. In inches, $m(\overline{CD}) = m(\overline{AB})$.

The measure of $\overline{AB}$ to the nearest inch is 2. We write this, $m(\overline{AB}) = 2$. The measure of $\overline{CD}$ to the nearest inch is also 2; $m(\overline{CD}) = 2$.

For the same two segments we may get a different measure if we use a different unit segment. It should be clear that if the unit is changed, the scale changes. Thus, if we decide to use the centimeter as our unit, the scale appears as in Figure 16-6 and Figure 16-8 shows that in centimeters $m(\overline{AB}) = 4$ and $m(\overline{CD}) = 6$. Now the measures do indicate that

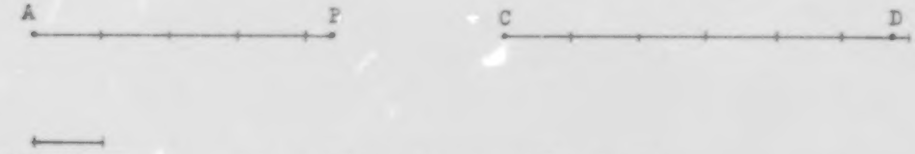


Figure 16-8. In cm., $m(\overline{CD}) > m(\overline{AB})$

there is a difference in the lengths of the two segments. Notice that by using a smaller unit (the centimeter) we are able to distinguish between the lengths of two non-congruent segments which in terms of a larger unit (the inch) have the same measure. If measurements of the same segment are made in terms of different units, the error in the measurements may be different since it is at most half the unit being used. Thus, if a segment is measured in inches the error cannot be more than half an inch, while if it is measured in tenths of an inch the error cannot be more than half of a tenth of an inch. As a result, if greater accuracy is desired in any measurement, a smaller unit should be used.

Sometimes it is more convenient to record a length of 31 inches as 2 feet 7 inches. Whenever a length is recorded using more than one unit, it is understood that the accuracy of the measure is indicated by the smallest unit named. A length of 4 yd. 2 ft. 3 in. is measured to the nearest inch. That is, it is closer to 4 yd. 2 ft. 3 in. than it is to either 4 yd. 2 ft. 2 in. or 4 yd. 2 ft. 4 in. A length of 4 yd. 2 ft. is interpreted to mean a length closer to 4 yd. 2 ft. than to 4 yd. 1 ft. or 4 yd. 3 ft. However, if this segment were measured to the nearest inch we would have to indicate this by 4 yd. 2 ft. 0 in. or 4 yd. 2 ft. (to the nearest inch). There is a very real difference in the accuracy of these measurements. When the measurement is made to the nearest foot, the interval within which the length may vary is one foot, when the measurement is made to the nearest inch, the interval within which the length may vary is one inch. This is because the end of the last unit counted may lie up to a half a unit on either side of the end of the segment.

A very important property of line segments is that any line segment may be measured in terms of any given unit. This means that no matter how small the unit may be, there is a whole number $\underline{n}$, such that if we lay off the unit $\underline{n}$ times along $\overline{AB}$ starting at A we will cover $\overline{AB}$ completely; that is, a point will be reached that is at the point B or beyond the point B on $\overrightarrow{AB}$.

The length of a line segment is a property of the line segment which we may measure in terms of different units. Theoretically, two segments have the same length if, and only if, they are congruent. We run into trouble thinking and talking about length because, in practice, measurement of length is made in terms of units and, as we saw above, two lines which are really different in length may both be said quite truly to have length 2 inches to the nearest inch. See Figure 16-7.

A vivid illustration of this trouble will emerge if we think about an application of linear measurement to the calculation of the perimeter of a polygon. By definition:

> The perimeter of a polygon is the length of the line segment which is the union of a set of non-overlapping line segments congruent to the sides of the polygon.

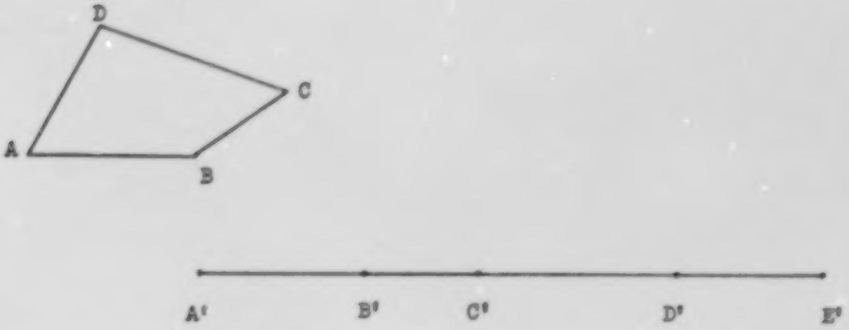


Figure 16-9.

Thus the perimeter of polygon ABCD is the length of $\overline{A'E'}$ where $\overline{A'E'}$ is the union of $\overline{A'B'}$, $\overline{B'C'}$, $\overline{C'D'}$ and $\overline{D'E'}$ which are respectively congruent to $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$. If we put pins at points A, B, C and D and stretch a taut thread around the polygon from A back to A, when we straighten out our thread we will have a model of a segment congruent to $\overline{A'E'}$.

The length of $\overline{A'E'}$, we know intuitively, is the sum of the lengths of the four segments when we consider length as an intrinsic property of segments. But, when we talk about lengths as measured in terms of certain units we may run into the following situation:

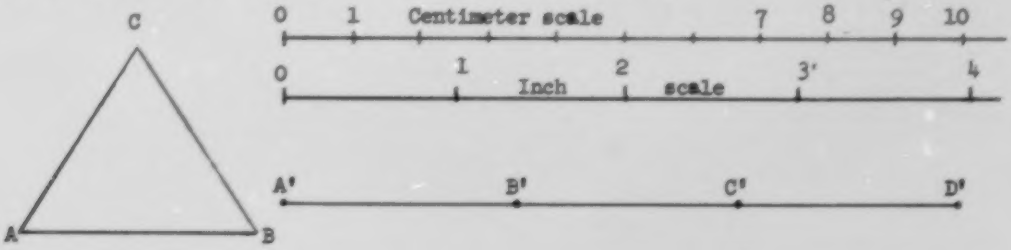


Figure 16-10.

To the nearest centimeter m($\overline{AB}$) = m($\overline{BC}$) = m($\overline{CA}$) = 3. $\overline{AB} \cong \overline{A'B'}$, $\overline{BC} \cong \overline{B'C'}$, $\overline{CA}$ = $\overline{C'D'}$ but m($\overline{A'D'}$) = 10. This is because to the nearest millimeter m($\overline{AB}$) = m($\overline{BC}$) = m($\overline{CA}$) = 33, and to the nearest millimeter m($\overline{A'D'}$) = 99, and to the nearest centimeter this means m($\overline{A'D'}$) = 10. Even if we measure our segments to the nearest inch we find m($\overline{AB}$) = m($\overline{BC}$) = m($\overline{CA}$) = 1 and we would expect the measure of the perimeter to be 3. But we find m($\overline{A'D'}$) = 4. This reminds us again that the measure of a length is always, at best, an approximation and approximation errors may accumulate to cause real trouble. The best we can say is to be aware of this possibility whenever in your problems you are dealing with numbers which turn up from measurement processes.

Problems

3. Two children are asked to determine the length and width of a crate; one is given a ruler with units marked in feet, the other a ruler with units marked in inches. The first says the crate is 3 feet long and 2 feet wide; the second says it is 40 inches by 28 inches. Explain why they could both be right.
4. Both children are asked to find the perimeter of the crate. The first one says 10 feet, the second says 136 inches. A string is then passed around the crate, stretched out and the children are asked to measure the string to find the perimeter. This time the first one says 11 feet, the second one 137 inches. Which results are correct? Explain the discrepancy between the results.

The Measure of an Angle

Just as we think of every line segment as having a certain exact length, so too we think of every angle as having a certain exact size, even though this size can be determined only approximately by measuring a chalk or pencil drawing of it.

Let us examine a process for measuring angles. As with linear measure we need to devise a way to assign a number as the measure to each angle. The first step is to select an arbitrary angle to serve as a unit and agree that its measure is the number 1. In Figure 16-11 we take $\angle XYZ$ as our unit.

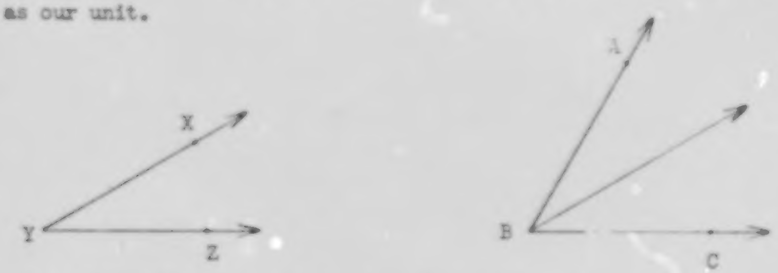


Figure 16-11. A unit $\angle$ and an angle of measure 2.

Now we can conceive of forming an angle $\angle ABC$ by laying off the unit twice about a common vertex B as suggested in the same figure. We say that in terms of the unit $\angle XYZ$ the measure of $\angle ABC$ is 2. We write:

In terms of unit $\angle XYZ$, $m(\angle ABC) = 2$.

In similar fashion we can conceive of forming angles whose measures are 3 or 4 and so forth until we come to an angle whose interior is nearly a half-plane as shown in Figure 16-12.

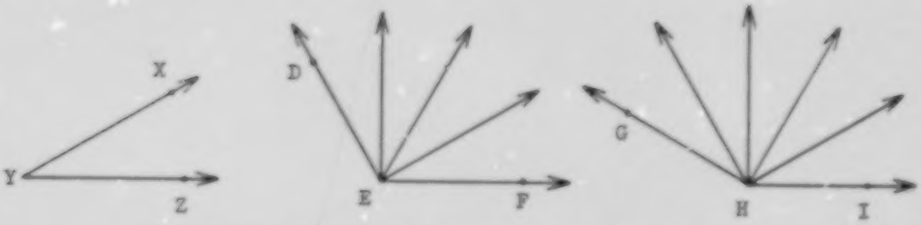


Figure 16-12. In terms of unit ∠ XYZ,
m(∠ DEF = 4), m(∠ GHI) = 5.

We can also conceive of an angle such that our unit ∠ XYZ will not fit into it a whole number of times. In Figure 16-13 we have an ∠ ABC in which if we start at $\overrightarrow{BC}$, the unit ∠ XYZ can be laid off 2 times about B without quite reaching $\overrightarrow{BA}$, that is with $\overrightarrow{BQ}$ in the interior of the angle, though if we were to lay it off 3 times we would arrive at a ray, call it $\overrightarrow{BP}$, which is well beyond $\overrightarrow{BA}$.

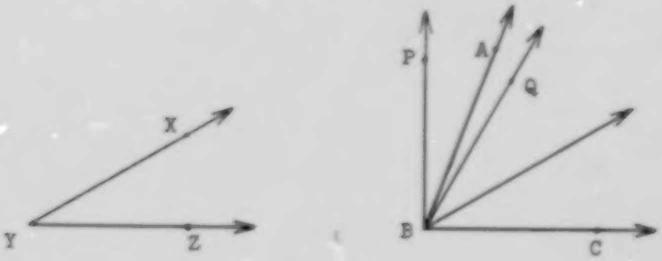


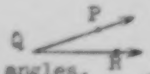
Figure 16-13. In terms of unit ∠ XYZ,
m(∠ ABC) = 2.

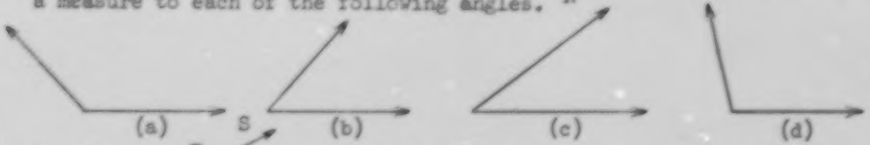
What can be said about the size of ∠ ABC? Surely, it is greater than 2 units and less than 3 units. In Figure 16-13, we can also estimate by eye that the size of ∠ ABC is nearer to 2 units than to 3 units, so:

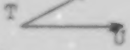
To the nearest unit in terms of unit ∠ XYZ
m(∠ ABC) = 2.

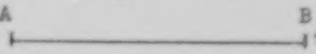
This is the best that can be done without considering fractional parts of units, or else shifting to a smaller unit. We always assign the measure so that the error is less than half the unit angle. But just as in measuring segments the measure of a specific given angle is almost always an approximation. A smaller error may be obtained by using a smaller unit but we can never be absolutely accurate. This sort of trouble can never be avoided. It is inherent in the approximations necessary for any measurement process. We just have to learn to live with it both now and later on when we study area and volume.

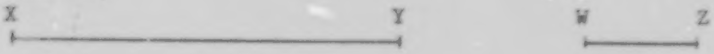
Problems

5. Make a careful tracing of the unit  and use it to assign a measure to each of the following angles.

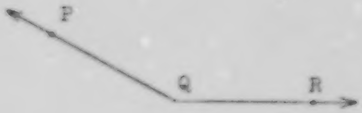


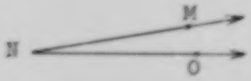
6. Use the unit  to assign a measure to the same set of angles.

•7. Use the unit  to assign a measure to $\overline{XY}$ and $\overline{WZ}$.



•8. Use the unit $\angle$ STU of Problem 6 to assign a measure to $\angle$ PQR and $\angle$ MNO.





Just as when we considered the measure of segments we found that a standard unit, a scale and a ruler were useful, we find the corresponding elements valuable in angle measure. The most common standard unit of angle measure is called a "degree." We write it in symbols as 1°. When we speak of the size of an angle, we may say its size is 45°, but if we wish to indicate its measure we must keep in mind that a measure is a number and say that its measure, in degrees, is 45. If we lay off 360 of these unit angles using a single point as a common vertex, then these angles, together with their interiors, cover the entire plane. Note that if $\angle$ ABC is a right angle, $m(\angle ABC) = 90$. A scale of degree units marked on a semi-circle is called a protractor.

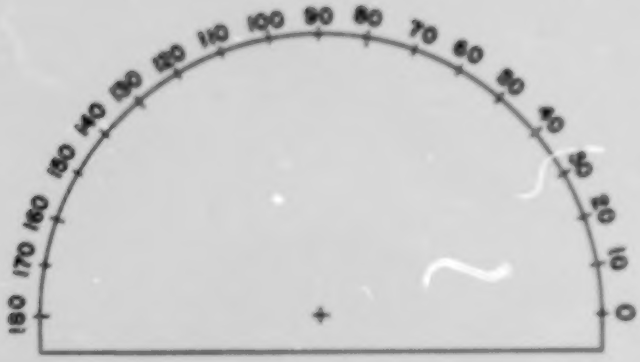


Figure 16-14. A protractor.

Even in ancient Mesapotamia the degree was used as the unit of angle measure. The selection of a unit angle which could be fitted into the plane (as above) just 360 times was probably influenced by their calculation of the number of days in a year as 360.

Problems

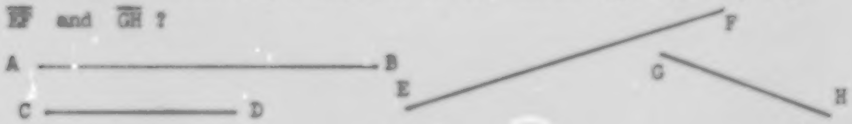
9. Use a protractor to assign a measure in degrees to each of the angles in Problems 5 and 8.
10. Use a centimeter scale to assign a measure to each of the segments in Problem 7.

Measurement of time is another example of the use of a standard unit such as an hour and of smaller units such as a minute or a second when greater accuracy is wanted. Nowadays scientific and engineering problems demand such accuracy that a common unit of length is an angstrom which is one hundred millionth of a centimeter and a micro-second which is a millionth of a second. These units are very small. Astronomers on the other hand for their purposes use very large units such as the "astronomical unit" which is the average distance of the sun from the earth, and the "light year" which is the distance travelled by light in one year travelling at an average rate of about 186,000 miles per second. But whatever units are used, it should be remembered that measurement is always approximate and answers are expressed to the nearest unit, whatever unit is being used. The decision as to which is the most appropriate unit always has to be made whether we consider the distance to the nearest galaxy of stars to the nearest light year, the distance from Washington to New York to the nearest mile, the diameter of an automobile engine piston to the nearest thousandth of an inch or the length of a wave of light to the nearest angstrom.

Measurement is the foundation stone of science and a connecting link between the physical world around us and mathematics.

Exercises - Chapter 16

1. Which of the following statements is true about segments $\overline{AB}$, $\overline{CD}$, $\overline{EF}$ and $\overline{GH}$?



   a. $\overline{AB} \cong \overline{CD}$
   b. $\overline{AB} < \overline{CD}$
   c. $\overline{AB} > \overline{EF}$
   d. $\overline{AB} \cong \overline{EF}$
   e. $\overline{GH} < \overline{CD}$
   f. $\overline{GH} \cong \overline{CD}$

2. A dog weighs 18 pounds.
   a. The unit of measure is ________.
   b. The measure is ________.
   c. The weight is ________.

3. A desk is 9 chalk pieces long.
   a. Its length is ________.
   b. Its measure is ________.
   c. The unit of measure is ________.

4. In which of the following sentences are standard units used?
   a. He is strong as an ox.
   b. Put in a pinch of salt.
   c. We drink a gallon of milk per day.
   d. The corn is knee high.
   e. I am five feet tall.

5. Convert the following measurements to the unit indicated.
   a. 17 cm. to mm.
   b. 3.4 m. to cm.
   c. 48 mm. to cm.
   d. 357 mm. to m.
   e. 93 cm. to m.
   f. 9.1 m. to mm.

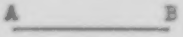
6. The measures of the sides of a triangle in inch units are 17, 15 and 13.
   - a. What are the measures of the sides if the unit is a foot?
   - b. What is the measure of the perimeter in inches? In feet?
   - c. Is there anything curious about your answer?
   - d. How do you explain it?

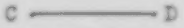
7. A man mailed 5 identical letters and found that he had to pay 8 cents postage apiece for a total of 40 cents. If he put them all in the same envelope his postage would only be 28 cents. Explain. Assume the postage rate to be 4¢ per ounce.

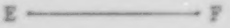
8. What is the measure of a right angle in degrees?

9. It is a fact that the sum of the measures of the angles of a triangle is 180. If the three angles of a triangle are congruent, what is the measure of each?

10. If two angles of a triangle are congruent and their measure is 30 what is the measure of the third angle?

11. Use  as a unit to measure the following segments.





Is $\overline{CD} \cong \overline{EF}$? Do your answers contradict each other? Explain.

*12. In Problems 7 and 8 of this chapter your answer should have $m(\overline{WZ}) = 0$ and $m(\angle MNO) = 0$. How is it possible to have this happen?

Solutions for Problems

1. a. 1; b. 2; c. 1; d. 2.
2. a. 2; b. 3; c. 1; d. 3. It should be noted how the measures differ.
3. 40 inches to the nearest foot is 3 feet since the error is less than $\frac{1}{2}$ foot. 28 inches to the nearest foot is 2 feet. Again the error is less than $\frac{1}{2}$ foot.

4. This problem involves the definition of perimeter of a polygon. Note that the perimeter is by definition the length of the segment which is congruent to the union of non-overlapping segments congruent to the sides. Thus the second method is the correct one for both children and the answers to the nearest unit are 11 feet and 137 inches. The first result comes from adding $3 + 2 + 3 + 2$ but each measure had an error of about 4 inches or $\frac{1}{3}$ of a foot and the accumulation of these leads to the result 10 feet which is, in fact, incorrect. The result 137 inches comes likewise because each side measured in inches had an error less than $\frac{1}{2}$ an inch but which accumulated to something near an inch. The difference between 11 feet and 137 inches is due to the fact that each child gives his answer correct to the nearest unit he is using.

5. a. 7; b. 2; c. 2; d. 5.

6. a. 5; b. 2; c. 1; d. 4.

* 7. $m(\overline{XY}) = 1$, $m(\overline{WZ}) = 0$

* 8. $m(\angle STU) = 5$, $m(\angle MNO) = 0$.

9. a. 133; b. 46; c. 36; d. 103
$m(\angle STU) = 152$; $m(\angle MNO) = 10$

10. $m(\overline{XY}) = 6$, $m(\overline{WZ}) = 2$

Chapter 17

FACTORS AND PRIMES

Introduction

We spoke before of the several aspects of mathematics which an elementary school mathematics program should contain. We noted that the conceptual aspect provides ideas by which a youngster can understand mathematics while the computational aspect gives him efficient ways of operating with the things of mathematics. In this chapter we propose to introduce some concepts involving only whole numbers, which, however, will prove to be very useful in developing computational procedures for dealing efficiently with fractions. Throughout this chapter "number" will mean the "whole numbers" 0, 1, 2, 3, 4, ...

Products and Factors

We have identified the word "product" with the "answer" to a multiplication problem. Thus 3×4 gives the product 12; 7×3 the product 21; 6×1 the product 6; and so on. Let us turn the problem around by considering what multiplications a given product could have come from. For example, what multiplications would give 10 as a product? Clearly they would be 1×10 or 2×5. (Of course 10×1 and 5×2 would also give the product 10 but because of the commutative property of multiplication these are not essentially different from those listed.) Similarly, 12 could be obtained as a product from 1×12, 2×6, or 3×4. The number 5 could come only from 1×5 and the number 13 could come only from 1×13.

Given any whole number b, we could list multiplications that give b as a product by listing the obvious product $b \times 1 = b$ and then asking, as a start, "Does some number times 2 give b as a product?" If the answer is "yes," i.e., if there is an n such that $n \times 2 = b$, we put this in our list of multiplications that give b as product and call both 2 and the number n factors of b. We continue then with the question, "Is there a number m such that $m \times 3 = b$?" If so, both 3 and m are factors of b. And so on. For example, 1, 3, 5 and 15 are all factors of 15 because each of these, with some other number, can be used in a multiplication problem which gives 15 as a product. Said more precisely, this all amounts to the following statement:

> a is a factor of b provided there is a number n such that $n \times a = b$. In this case n will also be a factor of b. (Remember that we are talking only about whole numbers.)

Problems*

1. Express each of the following numbers as products of two factors in several ways, or indicate that it is impossible to do so.

a.	18	c.	30
b.	6	d.	11

2. List all the numbers that would be called "factors"
   a. of the number 30,
   b. of the number 19,
   c. of the number 24.

Prime Numbers

We know that for every number b, $b \times 1 = b$; so that 1 and b are always factors of b. Also, to dispose of another special number, the fact that $n \times 0 = 0$ for every number n, means that every number is a factor of 0 and also that 0 is not a factor of any number except itself. That is, if zero is a factor in a multiplication, then zero is the only possible product no matter what other factors there are. Having said these things, we have said about all there is to say, in the present context, about 1 and 0 so that for the most part we will eliminate them from further consideration.

Now, for many numbers n it is not only true that 1 and n are factors but it is also the case that these are the only factors. For example, 1 and 5 are the only factors of 5; 1 and 3 the only factors of 3; 1 and 13 the only factors of 13; and so on. Such numbers have only themselves and 1 as factors. Numbers of this sort are of some interest to us so we will put them in a special classification as follows:

> Any whole number that has exactly two different factors (namely itself and 1) is a prime number.

Note that this definition excludes 1 from the set of primes, because 1 does not have two different factors, and that it excludes 0 from the set of primes because zero has more than two factors, as we discussed above.

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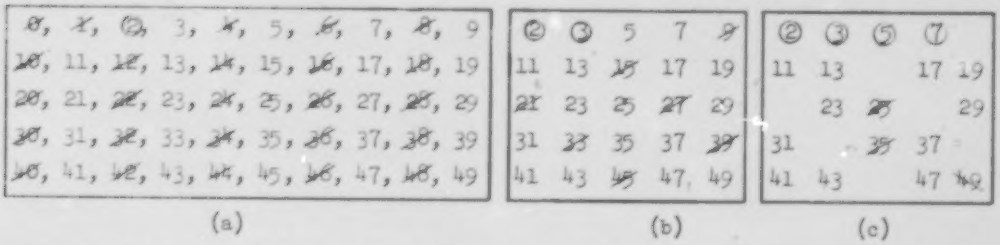
* Solutions for problems in this chapter are on page 216.

Zero and one are special numbers with special properties and we have the now defined "prime" numbers. All other numbers are put in a set as follows:

> All whole numbers other than 0, 1 and the prime numbers are called composite numbers.

The prime numbers have been a subject for mathematical speculation for centuries. Over 2,000 years ago the mathematician Eratosthenes invented an easy and straightforward way of sorting out the primes from the list of whole numbers. This method is known as "Eratosthenes Sieve" which describes the fact that it "lets through" only prime numbers. To get the primes less than 49 for example, we would list the numbers from 0 to 49 as shown in Figure 17-1a. Then cross out 0 and 1, since they are not primes; leave 2, since it is a prime, then cross out all other numbers with 2 as a factor since they cannot be primes. This is shown in Figure 17-1a by circling the 2 and crossing out multiples of 2 (4, 6, 8, 10, ...). Leave 3, since it is prime, but cross out all multiples of 3 that remain as shown in Figure 17-1b; that is, cross out 6, 9, 12, ... if they haven't already been crossed out. Of the numbers that remain, leave 5 and cross out its multiples; then leave 7 and cross out its multiples as shown in Figure 17-1c. All the numbers that now remain are prime numbers, as can be verified by attempting to factor them. Hence,



(a) (b) (c)

Figure 17-1. Use of Eratosthenes Sieve to get the primes between 0 and 49.

$$\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47\}$$

is the set of prime numbers between 0 and 49.

We might ask how carrying out this process only to 7 suffices to "sieve out" all the primes up to 49. We can see why this is so by observing that if we have a composite number less than 49, it can be expressed as a product m × n. If one of these two factors is 7 or some larger number, the other factor must be a number less than 7 because 7 times any number greater than 7 will give a product greater than 49. For example, $7 \times 8 = 56$, $7 \times 9 = 63$, etc. Since in such a product as $m \times n \leq 49$ one of the two numbers is 7 or less, crossing out multiples of 2, 3, 4, 5, 6 and 7 will take care of all such products less than 49, and hence all composite numbers less than 49.

Problem

3. a. List the prime numbers between 0 and 100.
   b. We know that $10 \times 10 = 100$. Pick several composite numbers less than 100, express each of them as a product of two factors in as many ways as you can, and verify that for any such composite number, one of these two factors will be less than 10.

Factoring and Prime Factorization

What we have said so far indicates that any composite number can be written as a product of two factors different from itself. This process is called factoring, and the result a factorization. Each of these factors may themselves be either prime or composite numbers; if either factor is composite it can be written as a product of two other numbers. If any of these are composite they can be factored. Such a process would end when each factor is a prime number. For example,

$$90 = 3 \times 30 = 3 \times (3 \times 10) = 3 \times 3 \times (2 \times 5) = 3 \times 3 \times 2 \times 5.$$

Of course, it is also true that $90 = 9 \times 10$, which is a different factorization. But if the process is continued we get:

$$90 = 9 \times 10 = (3 \times 3) \times (2 \times 5) = 3 \times 3 \times 2 \times 5.$$

Likewise:

$$90 = 6 \times 15 = (2 \times 3) \times (3 \times 5) = 2 \times 3 \times 3 \times 5.$$

And again:

$$90 = 2 \times 45 = 2 \times (5 \times 9) = 2 \times 5 \times (3 \times 3) = 2 \times 5 \times 3 \times 3.$$

Note that in each case the end result, containing only prime numbers, is the same except possibly for the order in which the factors are written. You can verify for yourself that this will be the case for any number you try.

This property, which can be proved although we will not do so here, is commonly stated as "The Fundamental Theorem of Arithmetic:"

> Every composite number can be factored as the product of primes in exactly one way except for the order in which the prime factors appear in the product.

Now, how do we get such a prime factorization? The most straightforward way would be simply to take our list of prime numbers and try each one in turn to see if it is a factor of the number in question. And since, for example, the question "Is 2 a factor of 28?" can be answered by seeing whether 2 divides 28 (without a remainder), we see that our test can be carried out by dividing as many times as possible by each prime number in turn. Hence to factor 28, 28 ÷ 2 = 14 so 2 is a factor; 14 ÷ 2 = 7, so 2 appears as a factor a second time; and 7 is itself a prime factor. In another form:

$$\begin{array}{r|l} 2 & 28 \\ 2 & 14 \\ & 7 \end{array}$$

. Hence the prime factorization of 28 is 28 = 2 × 2 × 7. Now look at the example in Figure 17-2 and make sure you see why the prime factorization of 1092 is 1092 = 2 × 2 × 3 × 7 × 13.

Step	
1	$2 \lfloor\ 1092$
2	$2 \lfloor\ 546$
3	$3 \lfloor\ 273$
4	$7 \lfloor\ 91$
5	13

Steps 1 and 2: Since 1092 and 546 are "even" we know that 2 will divide them.

Step 3: Clearly 2 doesn't divide 273 so we try 3, which works.

Step 4: Three doesn't divide 91; nor does 5 (you would know this without trying since 91 doesn't end in 5 or 0); so try 7, which works.

Step 5: Thirteen is a prime number so we are finished.

Figure 17-2. Prime factorization process showing 1092 = 2 × 2 × 3 × 7 × 13.

It might seem that this process of prime factorization by trying each prime in turn would be lengthy for large numbers. But it turns out that trying the primes through 7, for example, takes care of the factorization of any number up to 49 (7 × 7), as we have seen; the primes through 13 dispose of numbers through 169 (13 × 13); and the primes through 50 (which have already been listed) will dispose of all numbers through 2500 (50 × 50). Where prime factors are repeated, even larger numbers are disposed of. In general, trying the primes through p in the manner illustrated in Figure 17-2 will take care of the prime factorization of numbers less than or equal to the number $p \times p$.

Problems

4. Using the process illustrated above, find the prime factorization of each of the following:
   a. 105
   b. 75
   c. 320
5. How far, at most, in the list of primes would one need to go in order to find the prime factorization of
   a. 121
   b. 399
   c. 3600

The Greatest Common Factor of Two Numbers

We now know how to express any composite number as a product of prime factors. We can then use these prime factors to find the composite factors of a number. For example, $60 = 2 \times 2 \times 3 \times 5$ shows the prime factors of 60. The composite factors of 60 would be $2 \times 2 = 4$, $2 \times 3 = 6$, $2 \times 5 = 10$, $3 \times 5 = 15$, $2 \times 2 \times 3 = 12$, $2 \times 3 \times 5 = 30$ and, of course, $2 \times 2 \times 3 \times 5 = 60$. In other words, composite factors of a number can be found by forming the various products possible using the prime factors of the number.

The study of prime numbers and factorization of numbers has a number of facets and is of great interest in that branch of mathematics labelled the Theory of Numbers. Our concern at the moment, however, is to use the results we have obtained so far to develop just two ideas from this part of mathematics that prove to be useful in working with fractions. The first of these is the greatest common factor of two numbers, which we abbreviate g.c.f. For example, given the two numbers 12 and 18 the greatest common factor is simply the largest number which is a factor of 12 and at the same time a factor of 18. To see what it will be observe that:

the set of factors of $12 = \{1,2,3,4,6,12\}$ and
the set of factors of $18 = \{1,2,3,6,9,18\}$.

The common factors are the factors that appear in both sets, namely 1, 2, 3, 6; and of these the greatest common factor is, of course, 6. Two more examples are shown in Figure 17-3.

(a)

Set of factors of $20 = \{1,2,4,5,10,20\}$

Set of factors of $30 = \{1,2,3,5,6,10,15,30\}$

Set of common factors of 20 and $30 = \{1,2,5,10\}$

Greatest common factor of 20 and $30 = 10$

(b)

Set of factors of $18 = \{1,2,3,6,9,18\}$

Set of factors of $27 = \{1,3,9,27\}$

Set of common factors of 18 and $27 = \{1,3,9\}$

g.c.f. of 18 and $27 = 9$

Figure 17-3. G.c.f. of 20 and 30 and g.c.f. of 18 and 27.

Let us now take a harder example and see how we can find the g.c.f. Suppose we want the greatest common factor of 180 and 420. You might want this, for example, to know what number to divide numerator and denominator by in "reducing" the fraction $\frac{180}{420}$. First note that: $180 = (2 \times 3 \times 5) \times 6$ and $420 = (2 \times 3 \times 5) \times 2 \times 7$. We can pick out the "common block" of prime factors from each, namely $2 \times 3 \times 5$. Clearly, $2 \times 3 \times 5 = 30$ will be a common factor of both numbers. Furthermore, it will be the largest such common factor, since any larger common factor would have to be 30 times some number which is a common factor of both 180 and 420, and we have already used up all the common factors. To take another example, let us find the g.c.f. of 72 and 54. Examine the solution given in Figure 17-4 and see that you understand it:

54 = 2 × 3 × 3 × 3 and 72 = 2 × 2 × 2 × 3 × 3.
If we group the factors as follows:

54 = (2 × 3 × 3) × 3
72 = (2 × 3 × 3) × 2 × 2

we see that 2 × 3 × 3 is the "common block" of prime factors from each. Hence, 2 × 3 × 3 = 18 must be a common factor and since we have used up all the common prime factors, 18 must be the greatest such common factor. Hence, the g.c.f. of 54 and 72 is 18.

Figure 17-4. Finding the g.c.f. of 54 and 72.

It should be noted that for two prime numbers, say 13 and 29, the g.c.f. of the two numbers would be 1, for 1 is the only common factor of two prime numbers and so is certainly the greatest common factor.

Finding the greatest common factor of two numbers could be regarded as a binary operation, for, just as with other binary operations, we start with a pair of numbers and produce a third number. The table in Figure 17-5 that displays the results of such an operation up to the pair (10, 12) may look a bit strange to you, but if you examine it you will see that if you read it just as you do a multiplication table, it does give the correct results. For example, looking at the circled results, 4 is the g.c.f. of 4 and 8; 3 is the g.c.f. of 3 and 6; and 3 is the g.c.f. of 9 and 12.

g.c.f.	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	1	2	1	2	1	2	1	2	1	2
3	1	1	3	1	1	③	1	1	3	1	1	3
4	1	2	1	4	1	2	1	④	1	2	1	4
5	1	1	1	1	5	1	1	1	1	5	1	1
6	1	2	3	2	1	6	1	2	3	2	1	6
7	1	1	1	1	1	1	7	1	1	1	1	1
8	1	2	1	4	1	2	1	8	1	2	1	4
9	1	1	3	1	1	3	1	1	9	1	1	③
10	1	2	1	2	5	2	1	2	1	10	1	2

Figure 17-5. A "Table" for the g.c.f. "Operation."

Problems

6. Find from the table the g.c.f. of 8 and 12. Of 10 and 5.
7. Using the process illustrated in Figure 17-4 find the g.c.f.
   a. of 24 and 36.
   b. of 60 and 72.
   c. of 40, 48 and 72.

The Least Common Multiple of Two Numbers

The second idea to be considered here is that of the *least common multiple* of two numbers. Now the statement "4 is a *factor* of 12" also means that "12 is a *multiple* of 4." That is, the multiples of 4 are all the numbers with 4 as a factor, thus {0, 4, 8, 12, 16, 20, 24, 28, ...} is the set of multiples of 4. Likewise {0, 3, 6, 9, 12, 15, 18, 21, 24, 27, ...} is the set of multiples of 3. The common multiples of 4 and 3 are simply those numbers that appear in both sets, that is, in the intersection of the two sets. Hence {0, 12, 24, 36, ...} is the set of common multiples of 4 and 3. Zero is of no interest since it is a common multiple of *any* two numbers. The smallest common multiple, other than zero, we call the *least common multiple*; in the case of 4 and 3 it is 12. We abbreviate the least common multiple as l.c.m.

To find the l.c.m. of two numbers, say 60 and 108, we again begin by displaying their prime factorizations:

$$108 = 2 \times 2 \times 3 \times 3 \times 3$$
$$60 = 2 \times 2 \times 3 \times 5.$$

Since any multiple of 60 must, of course, contain all the factors of 60 and any multiple of 108 must contain all the factors of 108, any *common* multiple of the two numbers must contain all the factors contained in either 60 or 108. We could get a common multiple by simply taking 60×108 or, using their factors, $2 \times 2 \times 3 \times 5 \times 2 \times 2 \times 3 \times 3 \times 3$, but this would not be the *least* common multiple because we have more 2's and more 3's than we really need as factors for either 60 or 108. Let us use *only* the factors we *need*, namely, two 2's, three 3's and one 5. Hence $2 \times 2 \times 3 \times 3 \times 3 \times 5$ or 540 will be the l.c.m. of 60 and 108. To examine this result in more detail observe that 108 and 60 have a common block of factors $2 \times 2 \times 3$ so these must be in any common multiple. In addition to these we need the factors 3×3 to make up the 108. The number 5 is not in 108 and the l.c.m. must provide for it in order to be a multiple of 60. Hence, we take the common factors 2 × 2 × 3, throw in 3 × 3 to get 108, then 5 so as to have all the factors of 60. The result is:

$$\text{l.c.m. of } 60 \text{ and } 108 = \underbrace{\overbrace{2 \times 2 \times 3 \times 3 \times 3}^{108} \times 5}_{60} .$$

This construction guarantees a number that contains 108 as a factor, hence the number is a multiple of 108 and contains 60 as a factor, hence the number is a multiple of 60. Furthermore, since only essential factors have been used, this construction gives the smallest such common multiple of 108 and 60.

To go through another example, let us find the l.c.m. of 18 and 30. Since 18 = 2 × 3 × 3 and 30 = 2 × 3 × 5 our least common multiple must have as factors one 2, two 3's, and one 5 in order to contain the factors of both 18 and 30. Hence the l.c.m. of 18 and 30 will be 2 × 3 × 3 × 5 = 90. Again, you can easily verify that 90 is a multiple of 18 (namely 5 × 18) and a multiple of 30 (namely 3 × 30) and no smaller number is such a common multiple. In working the problem $\frac{5}{18} + \frac{7}{30}$, for example, 90 could be used as a least common denominator.

As with the g.c.f. we can observe that in working with the l.c.m. we start with a pair of numbers and produce a third number, called the l.c.m. Hence we could think of this as a binary operation, make a table of results, and investigate some of the properties of the operation. Completion of such a table is given as a problem below.

Problems

8. Find the l.c.m. of
   a. 12 and 18
   b. 24 and 36
   c. 9 and 9
   d. 13 and 11

9. Complete the 9 by 9 l.c.m. table begun below.

l.c.m.	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	2	6	4	10	6	14	8	18
3	3				15				
4	4				20				
5	5				5				
6	6				30				
7	7				35				
8	8				40				
9	9								

Some Other Properties Involving Primes

There are many properties of whole numbers, factors, primes, etc., beyond those discussed here. There are questions about numbers which can be stated simply but for which the answers are not known to anyone. Just for fun let's look at one statement we know is true, and one which may be true or may be false. We just don't know.

1. If $\underline{m}$ and $\underline{n}$ are any two numbers, the product of their g.c.f. and their l.c.m. = m × n.
2. Any even number (i.e., any number which is divisible by 2) greater than 4, can be expressed as the sum of two primes.

We know for sure that statement 1 is true. To demonstrate this let us suppose that $\underline{m}$ is some number such that it can be factored as 2 × 2 × (some other primes) and that $\underline{n}$ can be factored as 2 × 2 × 2 × (some other primes). Then the g.c.f. of $\underline{m}$ and $\underline{n}$ has exactly two 2's in it, since they are in the "common block" of factors; while the l.c.m. has exactly three 2's in it since it must contain all the factors of $\underline{n}$. Hence the product of the g.c.f. and the l.c.m. will have five 2's in it as factors. So will the product m × n. This same argument applied to each prime in turn gives the general statement 1.

No one knows whether the second statement is true or false. We know that 6 = 3 + 3, 12 = 5 + 7, 20 = 13 + 7, 24 = 11 + 13, etc. No one has yet found an even number which is not the sum of two primes, but, on the other hand, no one as of this writing (1963) has found a general proof of the statement.

Summary

We have considered factors, prime numbers, composite numbers, factoring, and prime factorizations. We have used these to study two ideas useful in computing with fractions, namely the greatest common factor and the least common multiple.

Let us consider one final example of the ideas studied in this unit by looking at 24 and 90:

$$24 = 2 \times 2 \times 2 \times 3$$
$$90 = 2 \times 3 \times 3 \times 5$$

$$\text{g.c.f. of } 24 \text{ and } 90 = 2 \times 3 = 6$$
$$\text{l.c.m. of } 24 \text{ and } 90 = 2 \times 2 \times 2 \times 3 \times 3 \times 5 = 360$$

$$(\text{g.c.f.}) \times (\text{l.c.m.}) = 6 \times 360 = 2160$$
$$m \times n = 24 \times 90 = 2160$$

$24 = 11 + 13$; $24 = 17 + 7$; $24 = 19 + 5$

$90 = 83 + 7$; $90 = 79 + 11$; $90 = 73 + 17$; etc.

Exercises - Chapter 17

1. List the primes between 100 and 150.

2. Express each of the following numbers as a product of two smaller numbers or indicate that it is impossible to do this:

   a. 12 c. 31 e. 8 g. 35 i. 39 k. 6 m. 82
   b. 36 d. 7 f. 11 h. 5 j. 42 l. 41 n. 95

3. Write a prime factorization of:

   a. 15 b. 30 c. 45 d. 13

4. Find a prime factorization of:

   a. 105 c. 64 e. 1000 g. 323
   b. 300 d. 311 f. 301

5. Find the greatest common factor in each of the following cases. (Use the results from Problems 3 and 4 wherever you can.)
   - a. 15, 45
   - b. 13, 30
   - c. 24, 36
   - d. 105, 300
   - e. 32, 48
   - f. 30, 64
   - g. 12, 24, 48
   - h. 40, 48, 72
   - i. 15, 30, 45
   - j. 20, 50, 100

6. a. What is the greatest common factor of 0 and 6?
   - b. What is the smallest common factor of 0 and 6?
   - c. What is the smallest common factor for any two whole numbers?

7. You have learned about operations with whole numbers; addition, subtraction, multiplication and division. In this chapter, we studied the operation of finding the greatest common factor. For this problem only, let us use the symbol " $\triangle$ " for the operation g.c.f. Thus for any whole numbers, a and b and c,

   $a \triangle b$ = g.c.f. for a and b

   or $a \triangle c$ = g.c.f. for a and c.

   Example: $12 \triangle 18 = 6$

   $9 \triangle 15 = 3$

   - a. Is the set of whole numbers closed under the operation $\triangle$?
   - b. Is the operation commutative; that is, does $a \triangle b = b \triangle a$?
   - c. Is the operation associative; that is, does $a \triangle (b \triangle c) = (a \triangle b) \triangle c$?

8. Find the least common multiple of the elements of the following sets. (Use the results of Problems 3 and 4 when appropriate.)
   - a. $\{2,3\}$
   - b. $\{3,5\}$
   - c. $\{3,7\}$
   - d. $\{5,7\}$
   - e. $\{15,30\}$
   - f. $\{30,45\}$
   - g. $\{2,13\}$
   - h. $\{7,11\}$
   - i. $\{105,300\}$
   - j. $\{11,13\}$
   - k. $\{2,3,5\}$
   - l. $\{64,300\}$

9. a. What is the least common multiple of 6 and 6?
   - b. What is the g.c.f. of 6 and 6?
   - c. What is the least common multiple of 29 and 29?
   - d. What is the g.c.f. of 29 and 29?
   - e. What is the least common multiple of a and a, where a is any counting number?
   - f. What is the g.c.f. of a and a, where a is any counting number?

10. a. Can the greatest common factor of a pair of whole numbers ever be the same number as the least common multiple of those whole numbers? If so, give an example.
    b. Can the greatest common factor of a pair of whole numbers ever be less than the least common multiple of those numbers? If so, give an example.
    c. Can the least common multiple for a pair of whole numbers ever be less than the greatest common factor of those whole numbers? If so, give an example.

Solutions for Problems

1. a. 3 × 6, 2 × 9, 1 × 18 (or 6 × 3, 9 × 2, etc.)
   b. 2 × 3, 1 × 6
   c. 2 × 15, 5 × 6, 3 × 10, 1 × 30
   d. 1 × 11 and 11 × 1 are the only such factorizations and they are not essentially different.
2. a. 1, 2, 3, 5, 6, 10, 15 and 30
   In more formal terms, the set of factors of 30 = {1,2,3,5,6,10,15,30}
   b. 1 and 19
   c. The set of factors of 24 = {1,2,3,4,6,8,12,24}.
3. In addition to the primes shown in Figure 17-1, we have 53, 59, 61, 71, 73, 79, 83, 89, 97.
4. a. $\begin{array}{r|l} 3 & 105 \\ \hline 5 & 35 \\ \hline & 7 \end{array}$ so 105 = 3 × 5 × 7
   b. 75 = 3 × 5 × 5
   c. 320 = 2 × 2 × 2 × 2 × 2 × 2 × 5
5. a. 11
   b. 19 (20 × 20 = 400 so the primes less than 20 will suffice)
   c. 59 (60 × 60 = 3600)
6. g.c.f. of 8 and 12 = 4; of 10 and 5 = 5

7. a. 12, since $24 = (2 \times 2 \times 3) \times 2$; $36 = (2 \times 2 \times 3) \times 3$
   b. 12
   c. 8

8. a. 36 b. 72 c. 9 d. $11 \times 13 = 143$

9.

l.c.m.	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	2	6	4	10	6	14	8	18
3	3	6	3	12	15	6	21	24	9
4	4	4	12	4	20	12	28	8	36
5	5	10	15	20	5	30	35	40	45
6	6	6	6	12	30	6	42	24	18
7	7	14	21	28	35	42	7	56	63
8	8	8	24	8	40	24	56	8	72
9	9	18	9	36	45	18	63	72	9

Chapter 18

INTRODUCING RATIONAL NUMBERS

Introduction

All our work with numbers up to this point has been with the set of whole numbers; we have pretended that they are the only numbers that exist and we have seen how they and their operations behave. Our number lines have been marked only at the points which correspond to whole numbers, leaving gaps containing many points that are not named. Linear measurement has been done to the nearest whole unit, which in some cases might give a quite inaccurate notion of "how big" something is, especially if an inappropriate unit is used, or it might assign the same number to two things that are clearly of different size. Also, using only whole numbers it is clear that many "division" problems cannot be worked (for example $3 \div 4$); that is, the whole numbers are not "closed" under the operation of division since division of whole numbers might not give a whole number answer.

Now the problem of naming points between those named by whole numbers on the number line; the problem of (almost) getting "closure" under division of whole numbers (we cannot divide by zero); and the need for getting greater accuracy in measurement are all problems that persuade us of the need to extend our number system to include more than the whole numbers. In the historical development of numbers the measurement problem was probably a significant motivation in forcing the extension of number systems to more sophistication than merely counting and numbering.

In our extension of the number system to include what we will call rational numbers (but which are frequently called "fractions") we will proceed much as we did with the whole numbers. That is, we will start right from scratch in this chapter developing physical models for such numbers and from these develop some concepts about them. The next few chapters will use this basis to develop procedures for comparing rational numbers, computing with them, and the like. To begin with we will assume only the work done so far with whole numbers and some intuitive notion of what is meant by the "area" of a region.

In setting up physical models for rational numbers we usually begin by fixing some "basic unit," for example, a segment, a rectangular region, a circular region, or a collection of identical things. This unit is then divided up into a certain number of "congruent" parts. These parts, compared to the unit, give us the basis for a model for rational numbers.

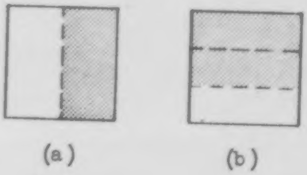

(a) (b)

Figure 18-1. Models for $\frac{1}{2}$ and $\frac{2}{3}$ using a square region as unit.

For example, let us identify as our base unit a square region and suppose this is divided into two congruent parts as shown in Figure 18-1a. We want to associate a "number" with the area of the shaded part of the square. Not only do we want a number, we want a name for this number, a numeral which will remind us of the two equal parts we have, of which one is shaded. The numeral is the obvious one, $\frac{1}{2}$, read "one-half." If our unit is divided into three congruent parts and if two of them are shaded, as in Figure 18-1b, the numeral $\frac{2}{3}$ reminds us that we are associating a number with two of three congruent parts of a unit. Observe that our numeral still uses notions expressible by whole numbers; that is, a basic unit is divided into three congruent parts with two of these considered.

In Figure 18-2, a rectangular region serves as the unit.

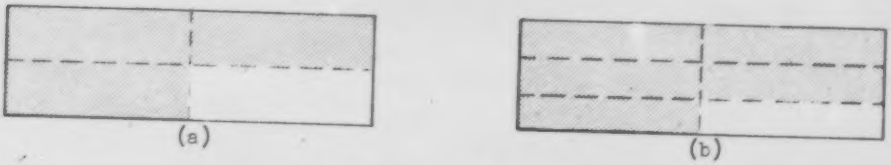
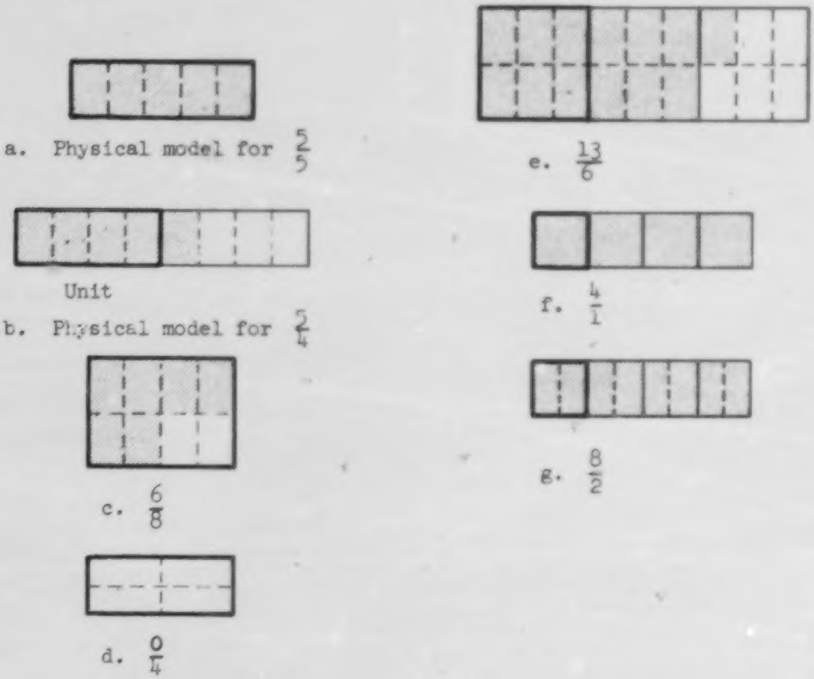

(a) (b)

Figure 18-2. Models for $\frac{3}{4}$ and for $\frac{5}{6}$.

The numeral $\frac{3}{4}$ expresses the situation pictured in Figure 18-2a, namely the unit region divided into four congruent regions, of which three are shaded. And, of course, the numeral $\frac{5}{6}$ expresses the situation represented by Figure 18-2b; the base unit divided into six congruent regions, of which five regions are shaded.

More complicated situations are represented in Figure 18-3. In each case the base unit is the rectangular region heavily outlined by solid lines. In some of these, the shaded region designates a region the same as or more than the base region, hence numbers equal to or greater than one. Thus Figure 18-3a shows the base unit divided into five parts, all of which are shaded. The numeral $\frac{5}{5}$ describes this model.



a. Physical model for $\frac{5}{5}$

b. Physical model for $\frac{5}{4}$

c. $\frac{6}{8}$

d. $\frac{0}{4}$

e. $\frac{13}{6}$

f. $\frac{4}{1}$

g. $\frac{8}{2}$

Figure 18-3. Models for various rational numbers using rectangular regions.

In Figure 18-3b, the unit region is divided into four congruent regions, and five such regions are shaded; the numeral $\frac{5}{4}$ describes this model. Examine the other situations illustrated in Figure 18-3 and verify that in each case the region shaded is indeed a model for the rational number written under it.

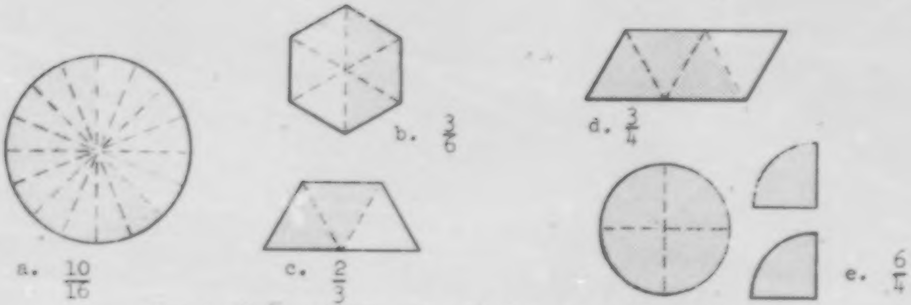


Figure 18-4. Models using regions of various shapes.

Regions of other shapes can also be used to represent rational numbers. Some such regions, with associated numerals, are pictured in Figure 18-4. In each case, you can verify that the model involves identification of a unit region, division of this region into congruent regions, and consideration of a certain number of these congruent regions.

Problems *

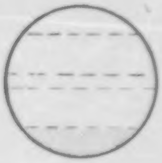
1. Draw models for:

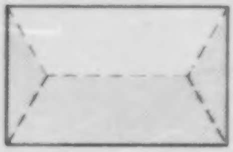
   a. $\frac{2}{3}$ d. $\frac{12}{5}$

   b. $\frac{4}{6}$ e. $\frac{7}{7}$

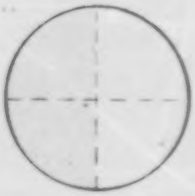
   c. $\frac{3}{2}$ f. $\frac{0}{6}$

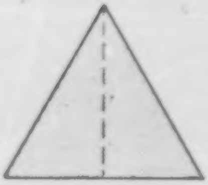
2. Why are the following pictures not good models for rational numbers?

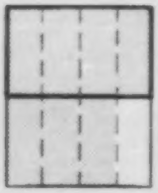

(a)

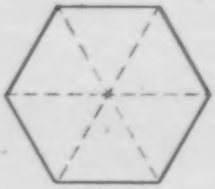

(b)

3. What numbers do the following models illustrate?


(a)


(b)


(c)


(d)

* Solutions for the problems in this chapter are found on page 228.

Number Line Models for Rational Numbers

Another standard physical model for the idea of a rational number uses the number line. If, as in Chapter 16, we have a ruler marked only in units, we cannot make certain types of useful measurement. We would like to be able to divide the unit intervals into equal parts. This would give us points between the unit intervals and we would like to have numbers associated with these parts.

The way we locate new points on the ruler parallels the procedure we followed with regions. We mark off a unit segment, then divide it into congruent segments. We then count off these parts. Thus, in order to locate the point corresponding to $\frac{1}{2}$, we mark off the unit segment into 2 congruent parts and count off 1 of them. (See Figure 18-5.) This point corresponds to $\frac{1}{2}$.

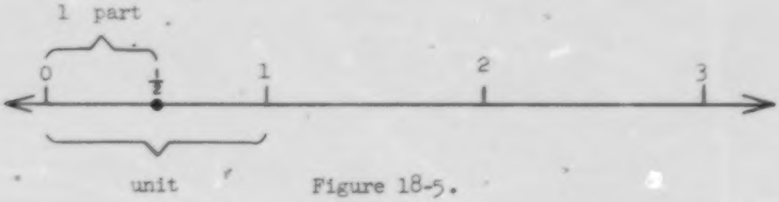


Figure 18-5.

In like manner, to locate $\frac{5}{4}$, we divide a unit interval into 4 congruent parts and count off 5 of these parts. We have now located the point which we associate with $\frac{5}{4}$ (Figure 18-6).

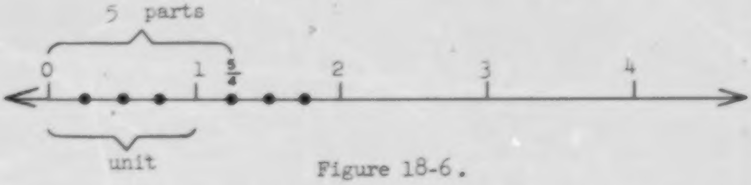


Figure 18-6.

Once we have this method in mind, we see that we can associate a point on the number line with all such symbols as $\frac{3}{4}$, $\frac{5}{8}$, $\frac{9}{4}$, etc., as illustrated in Figure 18-7.

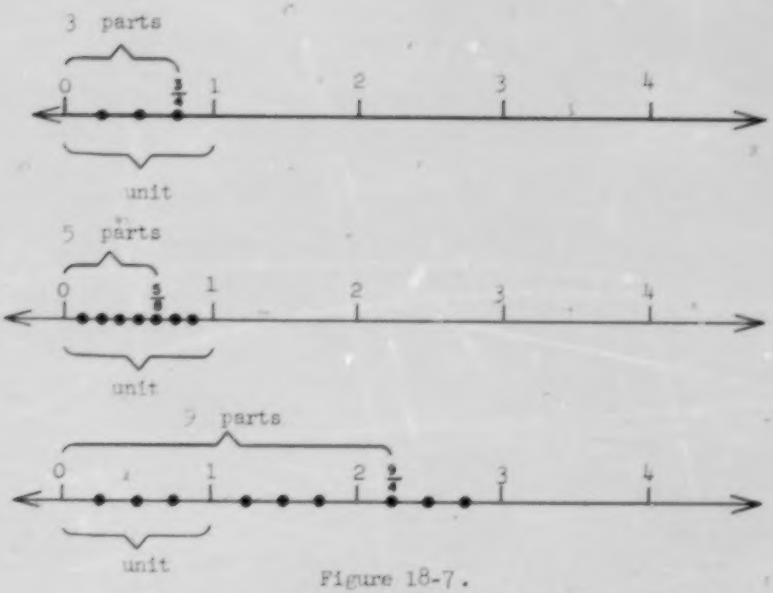


Figure 18-7.

Problem

4. Locate the point associated with each of the following on a separate number line.

a. $\frac{0}{1}$

b. $\frac{3}{4}$

c. $\frac{3}{5}$

d. $\frac{5}{5}$

e. $\frac{7}{4}$

f. $\frac{8}{8}$

Some Vocabulary and Other Considerations

The numbers for which our regions and segments are models are called rational numbers. The particular numeral form in which they are often expressed is called a fraction. We have here again the distinction between a number and numerals for this number. In general the "fractional form" $\frac{a}{b}$ represents a "rational number" provided a is a whole number and b is some whole number other than zero, that is, a counting number. Referring to our models, we see that b, the denominator, always designates how many congruent parts our unit has been divided into; while a, the numerator, indicates how many of these congruent parts are being used. One of several reasons why the denominator is never zero is that it would be nonsense to speak of a unit as being divided into zero parts; it surely cannot be divided into fewer than one part.

Figure 18-8 shows a number line on which we have located points corresponding to $\frac{0}{1}$, $\frac{1}{1}$, $\frac{2}{1}$, $\frac{3}{1}$, etc.; one on which we have located points corresponding to $\frac{0}{2}$, $\frac{1}{2}$, $\frac{2}{2}$, $\frac{3}{2}$, etc.; one on which we have located points corresponding to $\frac{0}{4}$, $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$, $\frac{4}{4}$, $\frac{5}{4}$, etc.; and one on which we have located points corresponding to $\frac{0}{8}$, $\frac{1}{8}$, $\frac{2}{8}$, $\frac{3}{8}$, $\frac{4}{8}$, etc.

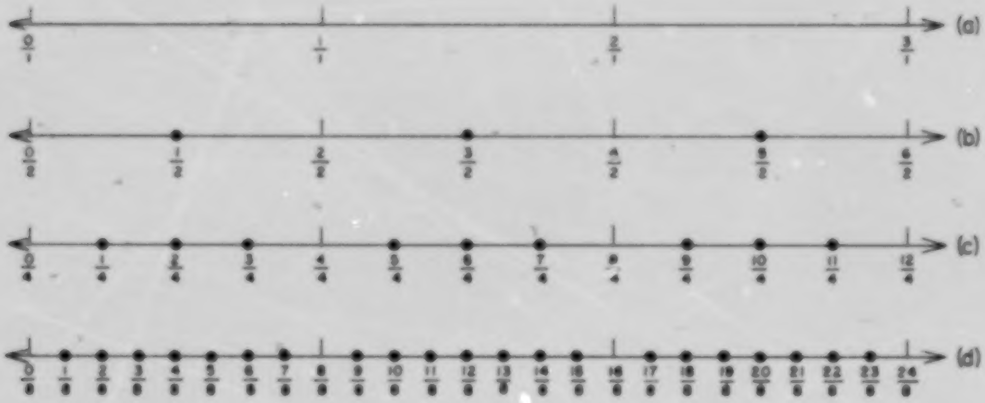

(a)
0/1 1/1 2/1 3/1
(b)
0/2 1/2 2/2 3/2 4/2 5/2 6/2
(c)
0/4 1/4 2/4 3/4 4/4 5/4 6/4 7/4 8/4 9/4 10/4 11/4 12/4
(d)
0/8 1/8 2/8 3/8 4/8 5/8 6/8 7/8 8/8 9/8 10/8 11/8 12/8 13/8 14/8 15/8 16/8 17/8 18/8 19/8 20/8 21/8 22/8 23/8 24/8

Figure 18-8. Points named on the number line.

As we look at the number lines in Figure 18-8, we see that it seems very natural to think of $\frac{0}{2}$, for example, as being associated with the zero point. For we are really, so to speak, counting off 0 segments. Similarly, it seems natural to locate $\frac{0}{1}$, $\frac{0}{4}$ and $\frac{0}{8}$ as indicated.

Now let us put the four number lines in Figure 18-8 together, as shown in Figure 18-9.

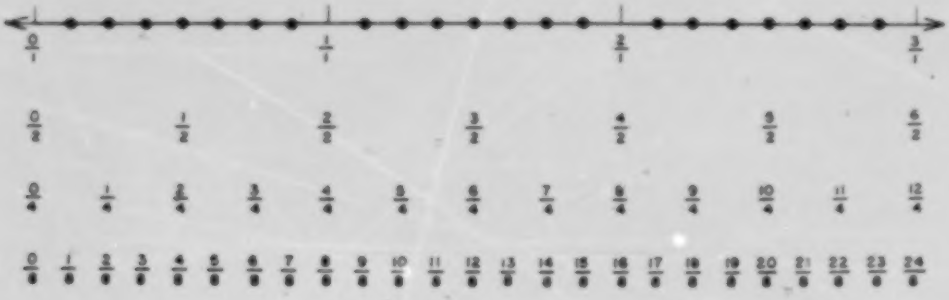

0/1 1/1 2/1 3/1
0/2 1/2 2/2 3/2 4/2 5/2 6/2
0/4 1/4 2/4 3/4 4/4 5/4 6/4 7/4 8/4 9/4 10/4 11/4 12/4
0/8 1/8 2/8 3/8 4/8 5/8 6/8 7/8 8/8 9/8 10/8 11/8 12/8 13/8 14/8 15/8 16/8 17/8 18/8 19/8 20/8 21/8 22/8 23/8 24/8

Figure 18-9 .

In other words, let us carry out on a single line the steps for locating in turn points corresponding to the rational numbers with denominator 1, with denominator 2, with denominator 4 and with denominator 8. When we do this we see, among other things, that $\frac{1}{2}$, $\frac{2}{4}$ and $\frac{4}{8}$ all correspond to the same point on the number line, or, in other words, are all names (numerals) for the same rational number. We see also that $\frac{0}{1}$, $\frac{1}{1}$, $\frac{2}{1}$, and so on, name the points we have formerly named with whole numbers. Furthermore we see that fractions such as $\frac{2}{2}$, $\frac{4}{4}$, $\frac{4}{2}$, $\frac{8}{4}$, and the like also name points that have formerly been named by whole numbers. We will consider such matters in more detail in the next chapter.

Exercises - Chapter 18

1. Using rectangular regions as your unit regions, represent each of the following by dividing up the units and shading in parts.

   a. $\frac{3}{4}$

   b. $\frac{2}{4}$

   c. $\frac{4}{4}$

   d. $\frac{5}{4}$

   e. $\frac{7}{5}$

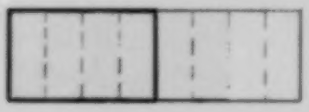
   f. $\frac{0}{3}$

   g. $\frac{9}{4}$

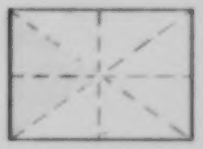
   h. $\frac{1}{7}$

2. Using unit segments on number lines, represent each of the fractions a - h of Exercise 1.

3. Most of the following figures are models for rational numbers. Some of them are not models because the unit has not been divided into congruent parts. For each one that is a proper model, give the rational number which is pictured. Which ones are not appropriate models?



(a)



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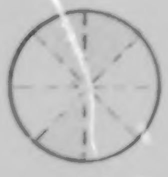
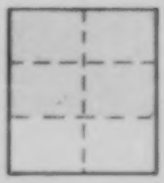


(c)



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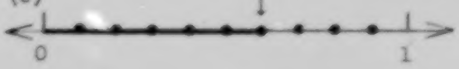

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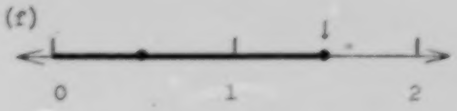

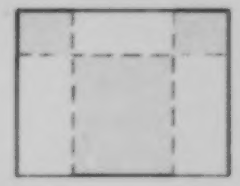

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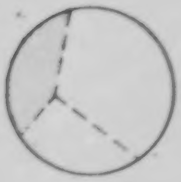
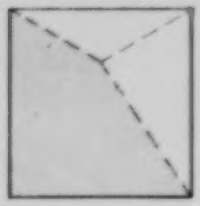
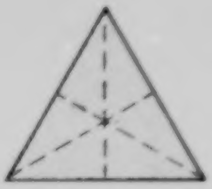


(h)

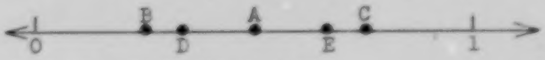


(i)



(j)

4. Consider the points labeled A, B, C, D and E on the number line:



a. Give a fraction name to each of the points.

b. Is the rational number located at point B less than or greater than the one located at D? Explain your answer.

c. In terms of the marks on this number line, what two fraction names could be assigned to the point A?

5. Interpret on the number line the following:

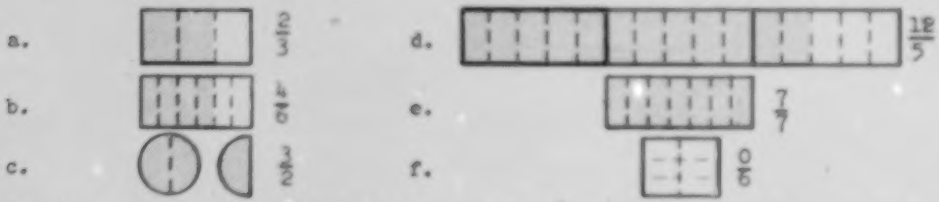
a. $\frac{20}{5} = 4$ b. $\frac{20}{4} = 5$ c. $\frac{23}{5} = 4\frac{3}{5}$

6. Show on the number line the equality:

$$\frac{2}{8} = \frac{3}{12}$$

Solutions for Problems

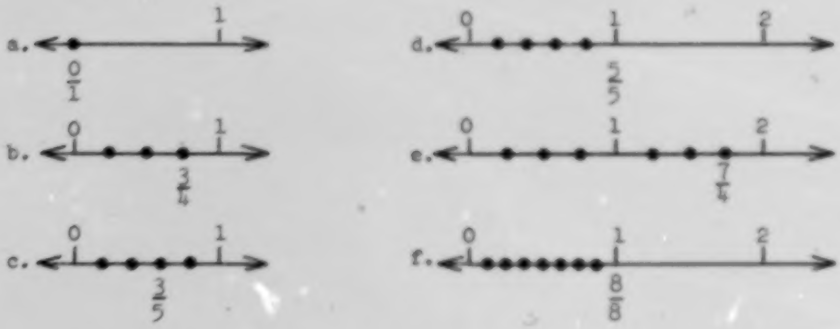
1. Many models will work here, these are illustrative only.


a.
2/3
b.
4/6
c.
3/2
d.
12/5
e.
7/7
f.
0/6

2. The figures are not good models because they are not divided into congruent parts.

3. a. $\frac{2}{4}$ b. $\frac{2}{2}$ c. $\frac{7}{4}$ d. $\frac{0}{6}$

4.


a.
1
0/1
b.
0
1
3/4
c.
0
1
3/5
d.
0
1
2
5/5
e.
0
1
2
7/4
f.
0
1
2
8/8

Chapter 19

EQUIVALENT FRACTIONS

Introduction

We have developed models for rational numbers from two different points of view; namely, unit regions and the number line. We have noted that fractions of the form $\frac{a}{b}$ name such numbers, with the counting number b designating how many congruent parts the unit region or segment is divided into and the whole number a designating how many of these congruent parts are being considered. It was noted briefly that each rational number has a variety of fractions that name it; for example, $\frac{1}{2}$, $\frac{2}{4}$, $\frac{3}{6}$ and $\frac{4}{8}$ all name the same number. It is the ramifications of this notion that we will explore in this chapter.

Recognizing the same rational number under a variety of disguises (names) and being able to change the names of numbers without changing the numbers are great conveniences in operating efficiently with rational numbers. Such an "addition" problem as $\frac{1}{4} + \frac{2}{3}$ is certainly worked out most efficiently by considering the equivalent problem $\frac{3}{12} + \frac{8}{12}$ -- equivalent because $\frac{1}{4}$ names the same number as $\frac{3}{12}$ and $\frac{2}{3}$ names the same number as $\frac{8}{12}$.

Equivalent Fractions in Higher Terms

Figure 19-1 illustrates a way of using our unit region model to show that $\frac{2}{3}$ and $\frac{8}{12}$ are equivalent fractions, that is, that $\frac{2}{3}$ and $\frac{8}{12}$ name the same number. First we select a unit region and divide it into three congruent regions by vertical lines as shown in Figure 19-1a. Figure 19-1b shows the shading of two of these regions to represent $\frac{2}{3}$. If we return now to our unit region and divide *each* of the former three congruent parts by horizontal lines into four congruent parts, we have the unit divided into $4 \times 3 = 12$ congruent parts, as shown in Figure 19-1c. If the unit divided in this way is now superimposed on the model for $\frac{2}{3}$, we get the model shown in Figure 19-1d, which shows that each of the two shaded regions in the model for $\frac{2}{3}$ is divided into four regions, giving

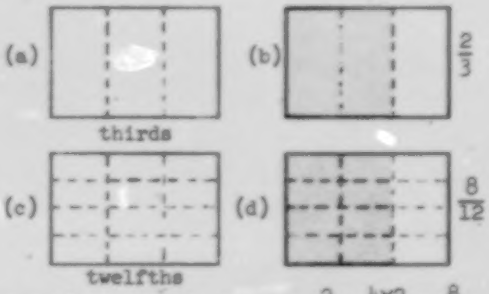


Figure 19-1. Model showing $\frac{2}{3} = \frac{4 \times 2}{4 \times 3} = \frac{8}{12}$.

$2 \times 4 = 8$ smaller congruent regions shaded. Hence the model showing 8 of 12 congruent parts represents the same number as the model showing 2 of 3 congruent parts.

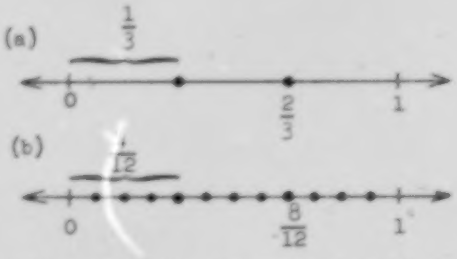


Figure 19-2. Number line model showing that $\frac{2}{3} = \frac{2 \times 4}{3 \times 4} = \frac{8}{12}$.

Figure 19-2 demonstrates this same equivalence. In Figure 19-2a, $\frac{2}{3}$ is shown by dividing the unit segment into 3 congruent parts and using two of these to mark a point. If each of the 3 congruent parts of the unit is now divided into 4 more congruent parts, the unit segment then contains $3 \times 4 = 12$ parts while the 2 original parts used to mark $\frac{2}{3}$ now contain $2 \times 4 = 8$ congruent parts, as shown in Figure 19-2b. Hence, the same point is named by $\frac{8}{12}$ as was formerly named by $\frac{2}{3}$. In both models, further subdivision of a unit results in the same sort of further subdivision of those parts of the unit used in representing the rational number.

To put this in more general terms, consider the fraction $\frac{a}{b}$ where b represents the number of parts a unit has been divided into and a the number of these parts marked in the model. If each of the b parts is further subdivided into k congruent parts, the unit then contains $b \times k$ congruent parts. At the same time, each of the a parts is further subdivided into k parts so that there will be $a \times k$ smaller congruent parts marked in the model. Hence, $\frac{a \times k}{b \times k}$ represents the same number as $\frac{a}{b}$ formerly did. Symbolically:

$$\frac{a}{b} = \frac{a \times k}{b \times k}$$

Hence $\frac{3}{4} = \frac{3 \times 5}{4 \times 5} = \frac{15}{20}$; $\frac{2}{3} = \frac{2 \times 20}{3 \times 20} = \frac{40}{60}$, and so on. Such a process enables us to express any fraction as an equivalent fraction "in higher terms," to use the usual terminology.

Problems*

1. Draw both a unit region model and a number line model to illustrate that $\frac{2}{3} = \frac{4}{6}$.

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* Solutions for the problems in this chapter are on page 240.

2. Supply the missing numbers in each of the following.

a. $\frac{3}{5} = \frac{3 \times}{5 \times} = \frac{24}{40}$ b. $\frac{7}{8} = \frac{\quad}{32}$ c. $\frac{\quad}{12} = \frac{14}{24}$

3. Specify the "k" used in each case to change the first fraction to the second.

a. $\frac{7}{13} = \frac{7 \times k}{13 \times k} = \frac{28}{42}$; $k =$ ____

b. $\frac{14}{16} = \frac{42}{48}$; $k =$ ____

c. $\frac{3}{7} = \frac{63}{147}$; $k =$ ____

Equivalent Fractions in "Lower Terms"

Expressing a fraction in lower terms (often called "reducing" fractions) is simply reversing, or undoing, the process used to express fractions in higher terms. For example, $\frac{2}{3} = \frac{2 \times 10}{3 \times 10} = \frac{20}{30}$ and, undoing this process, $\frac{20}{30} = \frac{20 \div 10}{30 \div 10} = \frac{2}{3}$. Similarly, $\frac{10}{4} = \frac{10 \div 2}{4 \div 2} = \frac{5}{2}$, $\frac{12}{18} = \frac{12 \div 3}{18 \div 3} = \frac{4}{6}$; $\frac{147}{3} = \frac{147 \div 3}{3 \div 3} = \frac{49}{1}$ and so on. In general:

> If both a and b are divisible by counting number k, then $\frac{a}{b} = \frac{a \div k}{b \div k}$.

In this case we say that the fraction $\frac{a}{b}$ has been reduced to "lower terms." It should be noted that while it is always possible to change a fraction to an equivalent one in higher terms with denominator any desired multiple of the original denominator, it is not always possible to "reduce" a fraction using a specified divisor, since one cannot always divide a counting number by a counting number. For example, $\frac{4}{6}$ can be reduced using 2 as a divisor, but not by using 3, while $\frac{3}{5}$ cannot be reduced at all. We sometimes say that a fraction which cannot be reduced, such as $\frac{1}{3}$, $\frac{4}{7}$, etc., is in simplest form or lowest terms (not to be confused with "lower" terms).

Putting fractions in "lowest terms" or "simplest form" is a convenient skill, but its importance has been overrated. The superstition that fractions must always, ultimately, be written in this form has no mathematical basis, for only different names for the same number are at issue. It is often convenient for purposes of further computation or to make explicit a particular interpretation to leave results in other than "simplest" form. However, where simplest form is desired we can proceed by repeated division in both numerator and denominator, or we can use the greatest common factor of both numerator and denominator as the k by which both should be divided. For, as you will recall from Chapter 17, the g.c.f. of two numbers is the largest number which is a factor of both numbers (or in other words, which divides both without remainders) and this is precisely what is required. The examples displayed in Figure 19-3 should be sufficient to illustrate both procedures for writing a fraction in an equivalent "simplest form."

a. $\frac{12}{20} = \frac{12 \div 2}{20 \div 2} = \frac{6}{10} = \frac{6 \div 2}{10 \div 2} = \frac{3}{5}$

a'. $12 = (2 \times 2) \times 3$
$20 = (2 \times 2) \times 5$

So the g.c.f. of 12 and 20 is the "common block" of factors $2 \times 2 = 4$.

$$\frac{12}{20} = \frac{12 \div 4}{20 \div 4} = \frac{3}{5}$$

b. $\frac{104}{260} = \frac{104 \div 2}{260 \div 2} = \frac{52}{130} = \frac{52 \div 2}{130 \div 2} = \frac{26}{65} = \frac{26 \div 13}{65 \div 13} = \frac{2}{5}$

b'. $$\begin{array}{r|l} 2 & 104 \\ 2 & 52 \\ 2 & 26 \\ & 13 \end{array} \qquad \begin{array}{r|l} 2 & 260 \\ 2 & 130 \\ 5 & 65 \\ & 13 \end{array}$$

So the g.c.f. is the "common block" $2 \times 2 \times 13 = 52$ and

$$\frac{104}{260} = \frac{104 \div 52}{260 \div 52} = \frac{2}{5}$$

Figure 19-3. "Simplest form" via repeated division and via use of the g.c.f. of numerator and denominator.

Problems

4. For each of the following, give one equivalent fraction in "higher terms" and give three equivalent fractions in "lower terms," including one in "lowest terms."

   a. $\frac{24}{36}$ b. $\frac{30}{60}$

5. Why would it not make sense to speak of a fraction raised to "highest terms"?

6. For each of the following, specify the g.c.f. of the numerator and denominator and use this g.c.f. to write the fraction in simplest form.

   a. $\frac{30}{45} =$ g.c.f. = ______
   b. $\frac{24}{36} =$ g.c.f. = ______
   c. $\frac{39}{52} =$ g.c.f. = ______

Order and Equivalence for Rational Numbers

Up to now we have focused pretty much on rational numbers with their various fractions one at a time. Let us look at the possible relations between two fractions, each of course representing a rational number. Recalling our work with whole numbers, we see that there are essentially three relations between any two numerals $\underline{n}$ and $\underline{m}$; either they are equivalent, that is, they name the same number; or the number $\underline{n}$ is "greater than" the number $\underline{m}$ (written as $n > m$); or the number $\underline{n}$ is "less than" the number $\underline{m}$ (written $n < m$). "Less than" and "greater than" tell the "order" such numbers come in when counting and hence are called "order relations."

A similar statement can be made about fractions:

> Given two fractions $\frac{a}{b}$ and $\frac{c}{d}$ one of these three things must be true:
>
> a. $\frac{a}{b} = \frac{c}{d}$, that is, they are equivalent fractions; or
>
> b. the rational number represented by $\frac{a}{b}$ is greater than that represented by $\frac{c}{d}$, in which case we write $\frac{a}{b} > \frac{c}{d}$; or
>
> c. the rational number named by $\frac{a}{b}$ is less than that named by $\frac{c}{d}$, in which case we write $\frac{a}{b} < \frac{c}{d}$.

Now we already know that two fractions are equivalent if one can be obtained from the other by multiplying both numerator and denominator by the same counting number. Hence $\frac{4}{8} = \frac{8}{16}$ since $\frac{8}{16} = \frac{4 \times 2}{8 \times 2}$. Given two fractions at random, however, this test may fail. For example, we know that $\frac{3}{6}$ and $\frac{4}{8}$ are equivalent (since both name the number one-half) yet there is no way of getting one from the other via multiplication of numerator and denominator. Nor would this test tell us anything about the pair of fractions $\frac{3}{6}$ and $\frac{68}{136}$. We could always put both on the same number line and see if the same point were named, or represent both in terms of the same unit region, but this would surely be a tedious business. What is needed, both here and in what follows, are efficient devices that depend primarily on previously learned notions involving whole numbers.

One way to handle the problem of telling when two or more fractions are equivalent is simply to reduce all of them to "lowest terms." For example, $\frac{3}{6} = \frac{3 \div 3}{6 \div 3} = \frac{1}{2}$; $\frac{4}{8} = \frac{4 \div 4}{8 \div 4} = \frac{1}{2}$; and, since the g.c.f. of 68 and 136 is 68, $\frac{68}{136} = \frac{68 \div 68}{136 \div 68} = \frac{1}{2}$; hence, $\frac{3}{6}$, $\frac{4}{8}$ and $\frac{68}{136}$ are equivalent. Reducing to lowest terms, of course, does involve only operations with whole numbers. Two other ways of testing equivalence will be given later on in this chapter.

Problems

7. Tell which of the following pairs of fractions are equivalent.

   a. $\frac{7}{14}, \frac{28}{56}$ b. $\frac{8}{6}, \frac{20}{15}$ c. $\frac{34}{51}, \frac{2}{3}$ d. $\frac{15}{20}, \frac{72}{96}$ e. $\frac{8}{12}, \frac{3}{4}$

8. In which of the following pairs could equivalence *not* be established by using $\frac{a}{b} = \frac{k \times a}{k \times b}$ to change one of them to have the same numerator and denominator as the other.

   a. $\frac{1}{2} = \frac{13}{26}$ b. $\frac{3}{6} = \frac{1}{2}$ c. $\frac{3}{6} = \frac{13}{26}$ d. $\frac{9}{12} = \frac{33}{44}$ e. $\frac{34}{51} = \frac{2}{3}$

Let us now consider the problem of designating the relations, less than or greater than, between two fractions that are not equivalent. (Of course, we really mean the relation between *the numbers which these fractions represent*, but it is awkward to keep saying this, so less precise terminology will be used on the assumption that by now the reader will know what is *really* meant.) Again, drawings and physical models can be used as in Figure 19-4 to illustrate the order by representing both fractions in terms of the same unit region or on the same number line, then noting which region is largest, or which point comes first on the number line. But this gets tedious and difficult for even mildly complicated cases, say the pair $\frac{9}{14}$ and $\frac{5}{8}$, so a standard computational procedure using only operations with whole numbers is indicated.

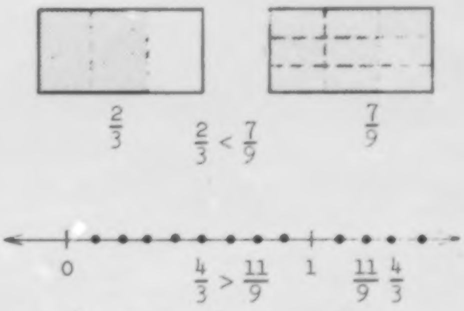


Figure 19-4. "Greater than" and "less than" models.

Now it is surely true that if two fractions have the same denominator, one can tell about order (or, for that matter, equivalence) merely by comparing numerators. So our problem is to find fractions equivalent to the ones in question, but with a "common denominator." To return to our example, $\frac{9}{14}$ and $\frac{5}{8}$, this means finding a common denominator that is a multiple of 14 and a multiple of 8, then changing the fractions to "higher terms" with this denominator in the usual way. Now there are many numbers which are multiples of both 8 and 14 (the product 8×14 is certainly one such number) and any of these numbers would serve our purpose, but it is usually most efficient to use the smallest such common multiple. This brings us to the "least common multiple" (l.c.m.) discussed in Chapter 17. Or, in other words, the problem of finding a "least common denominator" for two fractions is exactly equivalent to finding the "least common multiple" of their denominators. You will recall that this is done by factoring each number into primes, then constructing a number so that the factors of each will be included in the new number. In the present case, $14 = 2 \times 7$ and $8 = 2 \times 2 \times 2$ so the l.c.m. must have as factors three 2's and one 7; hence, the l.c.m. of 8 and 14 is $2 \times 2 \times 2 \times 7 = 56$. The complete problem in convenient computational form is displayed in Figure 19-5.

$$\frac{9}{14} = \frac{9}{2 \times 7} = \frac{2 \times 2 \times 9}{2 \times 2 \times 2 \times 7} = \frac{36}{56}$$

$$\frac{5}{8} = \frac{5}{2 \times 2 \times 2} = \frac{7 \times 5}{7 \times 2 \times 2 \times 2} = \frac{35}{56}$$

Hence: $\frac{9}{14} > \frac{5}{8}$

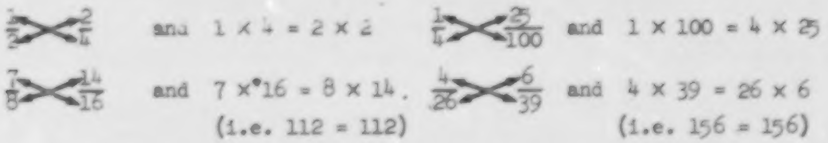
(Note that the l.c.m. of 8 and 14 is $2 \times 2 \times 2 \times 7$, so the factor 2×2 must be supplied in the first fraction, and the factor 7 in the second fraction.)

Figure 19-5. Finding the correct order relation for $\frac{9}{14}$ and $\frac{5}{8}$.

Observe that for purposes of computation it is convenient to factor the denominators, then construct fractions with the lowest common denominator by supplying the "missing" factors from the l.c.m. of the denominators.

Note that this same procedure would also take care of equivalence of two fractions, for if the two fractions had the same numerator when both were written with a common denominator, they would surely be equivalent. This is the second method of testing equivalence which was promised earlier in this chapter.

A third method for testing whether or not two fractions are equivalent is suggested by such examples as those in Figure 19-6.



$\frac{1}{2}$, $\frac{2}{4}$ and $1 \times 4 = 2 \times 2$ $\qquad$ $\frac{1}{4}$, $\frac{25}{100}$ and $1 \times 100 = 4 \times 25$

$\frac{7}{8}$, $\frac{14}{16}$ and $7 \times 16 = 8 \times 14$. (i.e. $112 = 112$) $\qquad$ $\frac{4}{26}$, $\frac{6}{39}$ and $4 \times 39 = 26 \times 6$ (i.e. $156 = 156$)

Figure 19-6. Examples showing that equivalent fractions have equal "cross products."

It is true, as the examples suggest, that if we have two equivalent fractions, the so-called cross products obtained by multiplying the first numerator times the second denominator and the first denominator times the second numerator give the same number. Furthermore, if we have two fractions for which it is not known whether or not they are equivalent, we can find out by computing and comparing these cross products. In other words:

Given two fractions $\frac{a}{b}$ and $\frac{c}{d}$, $\frac{a}{b} = \frac{c}{d}$
if and only if $a \times d = b \times c$.

A demonstration of why this should be so is given by considering how you might express any two fractions $\frac{a}{b}$ and $\frac{c}{d}$ as fractions with a common denominator. Now a common denominator, though possibly not the lowest common denominator, is surely the product, $b \times d$, of the two denominators. For example, a common denominator for $\frac{1}{3}$ and $\frac{1}{4}$ is $3 \times 4 = 12$, a common denominator for $\frac{1}{4}$ and $\frac{1}{8}$ is $4 \times 8 = 32$, and so on. If we express $\frac{a}{b}$ and $\frac{c}{d}$ with the common denominator $b \times d$, as shown below, the two numerators turn out to be $a \times d$ and $b \times c$; which are the cross products.

$$\frac{a}{b} = \frac{a \times d}{b \times d} \quad \text{and} \quad \frac{c}{d} = \frac{b \times c}{b \times d}$$

Since the denominators are the same, the fractions will be equivalent just in case these numerators are the same. A procedure similar to this is useful in working with ratios and proportions, as will be discussed in Chapter 24.

Problems

9. Use the procedure illustrated by Figure 19-5 to tell whether the first fraction of each pair given below is less than, equivalent to, or greater than the second fraction of the pair.

   a. $\frac{6}{14}, \frac{7}{16}$ $\qquad$ b. $\frac{6}{8}, \frac{9}{12}$ $\qquad$ c. $\frac{30}{63}, \frac{15}{28}$

10. By comparing the "cross products" tell which of the following pairs of fractions are equivalent.

   a. $\frac{3}{4}, \frac{36}{52}$ $\qquad$ b. $\frac{9}{20}, \frac{45}{100}$ $\qquad$ c. $\frac{143}{13}, \frac{1043}{103}$

A New Property of Numbers

Rational numbers are different in many ways from whole numbers. One such difference is apparent if we recall that for any whole number one can always say what the "next" whole number is and then ask, in a similar vein, what the "next" rational number is after any given rational number. For example, 4 is the next whole number after 3, 1069 is the next whole number after 1068, and so on, but what is the next rational number after $\frac{1}{2}$? If $\frac{2}{3}$ is suggested as the next one, we can observe that $\frac{1}{2} = \frac{6}{12}$ and $\frac{2}{3} = \frac{8}{12}$, so $\frac{7}{12}$ is surely between $\frac{1}{2}$ and $\frac{2}{3}$. Hence, $\frac{7}{12}$ has a better claim to being next to $\frac{1}{2}$ than does $\frac{2}{3}$. If it is then suggested that $\frac{7}{12}$ be regarded as the next number after $\frac{1}{2}$, we can observe that $\frac{1}{2} = \frac{12}{24}$ and $\frac{7}{12} = \frac{14}{24}$ so $\frac{13}{24}$ is closer to $\frac{1}{2}$ than is $\frac{7}{12}$. To carry this one step further, we can squelch anyone who suggests $\frac{13}{24}$ as being the next number after $\frac{1}{2}$ by pointing out that $\frac{1}{2} = \frac{24}{48}$ and $\frac{13}{24} = \frac{26}{48}$ so that $\frac{25}{48}$ is more nearly "next to" $\frac{1}{2}$ than is $\frac{13}{24}$. It is clear that this process could be carried on indefinitely and, furthermore, would apply no matter what rational number was involved. That is, we can never identify a "next" rational number after any given rational number. A similar argument would show that we cannot identify a number "just before" a given rational number.

A number line with a very large unit is shown in Figure 19-7 to illustrate the process we went through in searching for the number "next to" $\frac{1}{2}$.

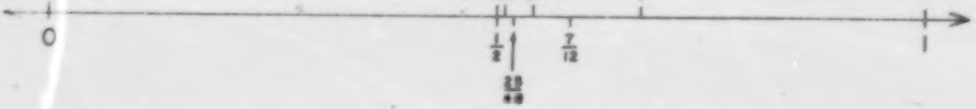
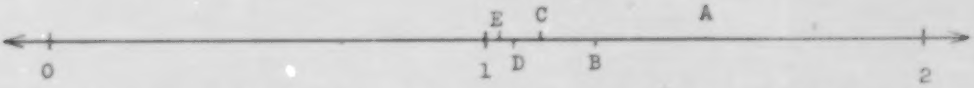


Figure 19-7. There is always a number between $\frac{1}{2}$ and any proposed "next" number.

Another way of expressing what we have been talking about is to say that between any two rational numbers, no matter how close together they are, there is always a third rational number; in fact, there are more rational numbers than we could count. Mathematicians sometimes describe this by saying that the set of rational numbers is "dense." The word is not important to us, but is descriptive of the packing of points representing rational numbers closer and closer together on the number line.

Problems

11. Name the rational numbers associated with the points A, B, C, D and E below, where A is halfway between 1 and 2, B halfway between 1 and A, etc.



12. How many numbers are there between 1 and the number associated with point E?

Summary

Any rational number can be represented by a number of different fractions, all of which are said to be equivalent. Any fraction can be changed to an equivalent fraction "in higher terms" by multiplying both numerator and denominator by the same counting number factor. Some fractions can be changed to equivalent fractions "in lower terms" by the inverse process, namely dividing both numerator and denominator by the same counting number. If a fraction has no common factors in its numerator and denominator it is said to be in "lowest terms" or "simplest form" and any fraction can be changed to this form by dividing numerator and denominator by their greatest common factor (g.c.f.). Given two fractions, "equivalence," "greater than" or "less than" can be specified by changing both fractions to fractions having a common denominator. For this purpose it is convenient to use the notion of least common multiple to find a least common denominator, especially since the construction of the l.c.m. via prime factorization of the denominators clearly indicates the multipliers that should be used in getting the equivalent fractions with the required denominator. As another way of testing equivalence of two fractions, we showed that $\frac{a}{b}$ and $\frac{c}{d}$ are equivalent if and only if their "cross products" are equal, that is, provided $a \times d = b \times c$.

Finally we have shown that between any two rational numbers, no matter how close, there are other rational numbers. Among other things this means that, unlike the whole numbers, one cannot identify the number that comes "just before" or "just after" a given rational number.

Even with this fairly detailed account we have not told the whole story with respect to numerals for rational numbers, as you know. For example, there are so called "mixed number" names such as $3\frac{1}{2}$, $1\frac{4}{5}$, etc., which do not have the fraction form. We will deal with these in the next chapter. There are also decimal names, such as .7 for $\frac{7}{10}$, .07 for $\frac{7}{100}$, and so on, that we will take up in Chapter 23.

Exercises - Chapter 19

1. Tell which of the following fractions are in "simplest form."

   $\frac{6}{12}$, $\frac{11}{4}$, $\frac{7}{12}$, $\frac{12}{13}$, $\frac{510}{513}$, $\frac{7}{412}$, $\frac{412}{7}$, $\frac{10}{12}$, $\frac{13}{26}$, $\frac{2}{3}$

2. It is true that in each fraction where a prime number appears in either the numerator or the denominator, the fraction is in lowest terms unless the other part of the fraction is a multiple of that prime.
   a. Which of the fractions in Exercise 1 demonstrates the truth of this statement?
   b. Explain why the statement must be true.

3. It is often possible to tell which of two fractions is largest just by having a clear conception of what is meant by the fractions. Thinking about them as they would appear on the number line is often a help in this. For each of the following, first make an educated guess about the order, then use some other means to check your guess.
   a. $\frac{1}{25}$, $\frac{1}{24}$
   b. $\frac{11}{24}$, $\frac{12}{26}$
   c. $\frac{7}{8}$, $\frac{5}{6}$
   d. $\frac{17}{34}$, $\frac{1}{2}$
   e. $\frac{13}{26}$, $\frac{9}{18}$

4. In each of the following, the order shown is the correct one.

   A. Under each pair write the two cross products $a \times d$ and $b \times c$ and insert the correct symbol "<", "=", ">" between these cross products. (The first one is done for you.)

   B. Examine your results and see if you can state a "cross products test" for the correct relation $<$ or $>$ between two fractions that are not equivalent.

a. $\frac{3}{6} < \frac{3}{5}$

$3 \times 5 < 6 \times 3$

b. $\frac{4}{5} < \frac{10}{11}$

c. $\frac{15}{8} < \frac{25}{12}$

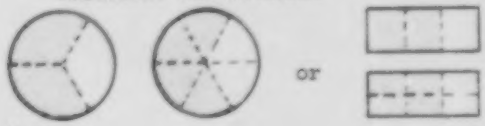
d. $\frac{9}{12} > \frac{8}{12}$

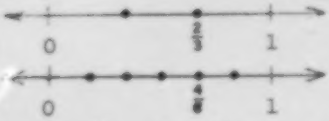
e. $\frac{13}{15} > \frac{2}{3}$

f. $\frac{337}{113} < \frac{167}{55}$

Solutions for Problems

1. For example:





2. a. 8 b. 28 c. 7

3. a. $k = 4$ b. $k = 3$ c. $k = 21$

4.

	Higher terms; many answers, e.g.:	Lower terms; any of these:	Lowest terms:
a.	$\frac{48}{72}$, $\frac{72}{108}$, $\frac{240}{360}$, etc.	$\frac{12}{18}$, $\frac{8}{12}$, $\frac{6}{9}$, $\frac{4}{6}$, $\frac{2}{3}$	$\frac{2}{3}$
b.	$\frac{60}{120}$, $\frac{180}{240}$, $\frac{240}{480}$, etc.	$\frac{15}{30}$, $\frac{10}{20}$, $\frac{6}{12}$, $\frac{5}{10}$, $\frac{3}{6}$, $\frac{1}{2}$	$\frac{1}{2}$

5. Since in $\frac{a}{b} = \frac{a \times k}{b \times k}$ $\underline{k}$ can be any counting number, there is no limit to how large the numerator and denominator can become.

6. a. $\frac{30 \div 15}{45 \div 15} = \frac{2}{3}$, g.c.f. = 15 c. $\frac{39 \div 13}{52 \div 13} = \frac{3}{4}$, g.c.f. = 13

   b. $\frac{24 \div 12}{36 \div 12} = \frac{2}{3}$, g.c.f. = 12

7. a, b, c, d

8. c. $\frac{3}{6} = \frac{13}{26} = \frac{1}{2}$ but $\frac{3}{6}$ cannot be changed directly to $\frac{13}{26}$.

   d. $\frac{9}{12} = \frac{33}{44} = \frac{3}{4}$ but no whole number multiplier $\underline{k}$ of 9 and 12 will give 33 and 44.

9. a. $\frac{6}{14} = \frac{6}{2 \times 7} = \frac{6 \times 8}{2 \times 7 \times 8} = \frac{48}{112}$

   $\frac{7}{16} = \frac{7}{2 \times 8} = \frac{7 \times 7}{2 \times 8 \times 7} = \frac{49}{112}$

   and $48 < 49$ so $\frac{6}{14} < \frac{7}{16}$

   b. $\frac{6}{8} = \frac{6}{2 \times 2 \times 2} = \frac{6 \times 3}{2 \times 2 \times 2 \times 3} = \frac{18}{24}$

   $\frac{9}{12} = \frac{9}{2 \times 2 \times 3} = \frac{9 \times 2}{2 \times 2 \times 3 \times 2} = \frac{18}{24}$

   so $\frac{6}{8} = \frac{9}{12}$

   c. $\frac{30}{63} = \frac{30}{3 \times 3 \times 7} = \frac{30 \times (2 \times 2)}{3 \times 3 \times 7 \times (2 \times 2)} = \frac{120}{252}$

   $\frac{15}{28} = \frac{15}{2 \times 2 \times 7} = \frac{15 \times (3 \times 3)}{2 \times 2 \times 7 \times (3 \times 3)} = \frac{135}{252}$

   so $\frac{30}{63} < \frac{15}{28}$

10. a. No, since $3 \times 52 \neq 4 \times 36$

    b. Yes, since $9 \times 100 = 20 \times 45 = 900$

    c. No, since $143 \times 103 = 14,729$ while $13 \times 1043 = 13,559$

11.

A	B	C	D	E
$\frac{3}{2}$ (or $1\frac{1}{2}$)	$\frac{5}{4}$ (or $1\frac{1}{4}$)	$\frac{9}{8}$ (or $1\frac{1}{8}$)	$\frac{17}{16}$ (or $1\frac{1}{16}$)	$\frac{33}{32}$ (or $1\frac{1}{32}$)

12. More than can be counted (actually "infinitely many").

Chapter 20

ADDITION AND SUBTRACTION OF RATIONAL NUMBERS

Introduction

We now have at hand the rational numbers and some physical models of them. We know something about ordering them with respect to "greater than" and "less than;" we can tell whether two fractions are equivalent; and we can change fractions to equivalent fractions in "lower terms" or "higher terms." The next natural step is to consider the ordinary binary operations with respect to rational numbers. That is, can we add, subtract, multiply, and divide using rational numbers? If so, how, and do the same properties apply as for addition, subtraction, multiplication and division of whole numbers? In considering these questions we must keep the following things in mind:

1. We have already observed, using the number line model, that such fractions as $\frac{3}{1}$, $\frac{5}{1}$, $\frac{10}{2}$ and the like, name points that are named by whole numbers; that such fractions as $\frac{1}{1}$, $\frac{2}{2}$, $\frac{13}{13}$, and, in general, $\frac{k}{k}$ all name the point named by 1; and that $\frac{0}{1}$, $\frac{0}{2}$, $\frac{0}{13}$, and, in general, $\frac{0}{k}$ all name the point named by 0. Since all whole numbers are also rational numbers--though of course not all rational numbers are whole numbers--we will want to make sure that the ordinary properties still apply. For example, "addition" should still be commutative and associative and the special properties of 0 and 1 should still be present.

2. A rational number has many names, hence we want to be sure that the results of a binary operation do not depend on the particular names we choose to use for the two numbers involved. In other words, we want $\frac{1}{6} + \frac{3}{6}$ to be the same number as $\frac{3}{18} + \frac{6}{12}$.

Incidentally, since we will be talking in the chapters that follow about rational numbers, the word "number" is to be taken until further notice to mean "rational number" unless specifically designated otherwise.

Addition of Rational Numbers

For two fractions with the same denominator, the matter of addition is easily disposed of and, for small denominators, physical models of the process are easy to draw. As one such model, we might think of a house, a school, and a store along a straight road as shown in Figure 20-1.

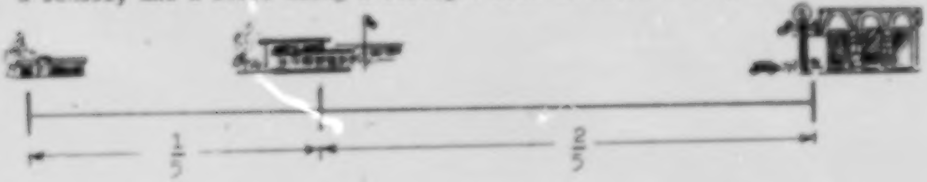
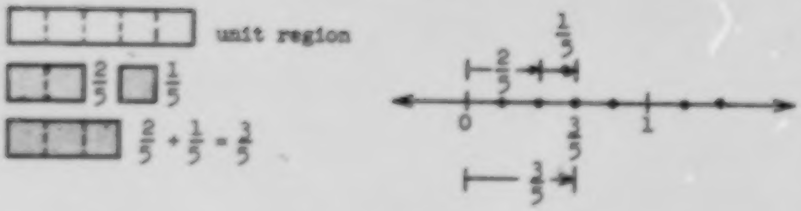


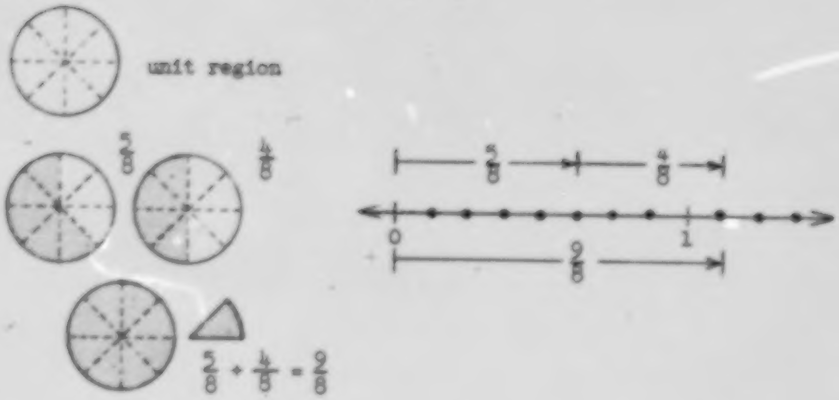
Figure 20-1.

If it is $\frac{1}{5}$ of a mile from the house to the school and $\frac{2}{5}$ of a mile from the school to the store, it is easy to see that it is $\frac{3}{5}$ of a mile from the house to the store. Other addition problems, with models that represent them, are shown in the Figure 20-2. In each case, both a number line model and a model using unit regions are given.

a. $\frac{2}{5} + \frac{1}{5} = \frac{3}{5}$


unit region
$\frac{2}{5}$
$\frac{1}{5}$
$\frac{2}{5} + \frac{1}{5} = \frac{3}{5}$
$\frac{2}{5}$
$\frac{1}{5}$
0
1
$\frac{3}{5}$
$\frac{3}{5}$

b. $\frac{5}{8} + \frac{4}{8} = \frac{9}{8}$


unit region
$\frac{5}{8}$
$\frac{4}{8}$
$\frac{5}{8}$
$\frac{4}{8}$
0
1
$\frac{9}{8}$
$\frac{5}{8} + \frac{4}{8} = \frac{9}{8}$

c. $\frac{6}{10} + \frac{5}{10} = \frac{11}{10}$

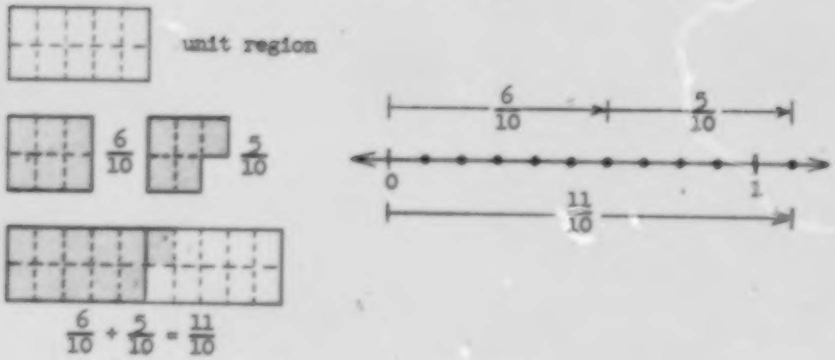

unit region
$\frac{6}{10}$
$\frac{5}{10}$
$\frac{6}{10}$
$\frac{5}{10}$
0
1
$\frac{11}{10}$
$\frac{6}{10} + \frac{5}{10} = \frac{11}{10}$

Figure 20-2. Models for addition of rational numbers.

From these examples we see that the way to add fractions having the same denominator is simply to add the numerators. Let us, then, make the following definition:

> Given two fractions $\frac{a}{b}$ and $\frac{c}{b}$ with the same denominator $b \neq 0$,
>
> $$\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b} .$$

Problems*

1. Fill in the blank spaces to make each example fit the pattern given by the definition above:

   a. $\frac{3}{4} + \frac{5}{4} = \frac{3 + \quad}{4} = \frac{8}{4}$

   b. $\frac{\quad}{16} + \frac{\quad}{16} = \frac{1 + 6}{16} = \frac{7}{16}$

   c. $\frac{4}{\quad} + \frac{5}{\quad} = \frac{4 + 5}{17} = \frac{9}{17}$

2. We know that $\frac{1}{8} = \frac{2}{16}$ and $\frac{3}{8} = \frac{6}{16}$. Show, using the definition and "reducing" your answers that $\frac{1}{8} + \frac{3}{8} = \frac{2}{16} + \frac{6}{16}$.

The way to deal with fractions that do not have the same denominator is, of course, simply to use equivalent fractions that do have the same denominator. In any such problem a number of choices are available. For example $\frac{1}{3} + \frac{1}{2}$ might become $\frac{2}{6} + \frac{3}{6}$; or $\frac{4}{12} + \frac{6}{12}$; or $\frac{12}{36} + \frac{18}{36}$; or what have you. It is usual, as you know, to find the least common denominator, that is, the l.c.m. of the denominators of the fractions, and use the fractions changed to that denominator in doing the addition. Since least common denominator was discussed at some length in connection with equivalence and ordering of fractions in Chapter 19, we will not deal with it further here.

The definition $\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}$ also gives us a way of dealing with so called "mixed numerals," such as $2\frac{1}{2}$. Such a numeral is read "two and one-half" and really designates the sum $2 + \frac{1}{2}$. Since $\frac{4}{2}$ is equivalent to the whole number 2, this becomes $\frac{4}{2} + \frac{1}{2}$ or, by our definition, $\frac{4 + 1}{2} = \frac{5}{2}$. Every such mixed numeral can be written as a fraction of the form $\frac{a}{b}$, and so we could also deal with the equivalence of mixed numeral and fraction names for rational numbers greater than 1. Observe that this also works in the other direction by considering our definition in reverse as $\frac{a + c}{b} = \frac{a}{b} + \frac{c}{b}$. For example:

---

* Solutions to problems in this chapter are on page 255.

$$\frac{5}{2} = \frac{4 + 1}{2} = \frac{4}{2} + \frac{1}{2} = 2 + \frac{1}{2} = 2\frac{1}{2}$$

$$\frac{5}{4} = \frac{4 + 1}{4} = \frac{4}{4} + \frac{1}{4} = 1 + \frac{1}{4} = 1\frac{1}{4}$$

$$\frac{14}{3} = \frac{12 + 2}{3} = \frac{12}{3} + \frac{2}{3} = 4 + \frac{2}{3} = 4\frac{2}{3}$$

In computing with mixed numerals one can either regroup using the commutative and associative properties (which we prove in the next section) in order to work with the whole number and fractional parts separately, or one can change the mixed numerals to ordinary fractions and compute using these fractions.

Examples of several addition computations, some in vertical form and some in horizontal form, are given in Figure 20-3. Of course, we do not ordinarily show all these details.

a. $\frac{7}{8} + \frac{2}{3} = \frac{21}{24} + \frac{16}{24} = \frac{21 + 16}{24} = \frac{37}{24} = \frac{24 + 13}{24} = \frac{24}{24} + \frac{13}{24} = 1 + \frac{13}{24} = 1\frac{13}{24}$

b. $4\frac{2}{3} + 1\frac{1}{6} = 4 + \frac{2}{3} + 1 + \frac{1}{6} = 4 + 1 + \frac{4}{6} + \frac{1}{6} = 5 + \frac{5}{6} = 5\frac{5}{6}$

c. $4\frac{2}{3} + 1\frac{1}{6} = (\frac{12}{3} + \frac{2}{3}) + (\frac{6}{6} + \frac{1}{6}) = \frac{14}{3} + \frac{7}{6} = \frac{28}{6} + \frac{7}{6} = \frac{28 + 7}{6}$

$= \frac{35}{6} = \frac{30 + 5}{6} = \frac{30}{6} + \frac{5}{6} = 5 + \frac{5}{6} = 5\frac{5}{6}$

d.
$$\begin{array}{ccc} 4\frac{2}{3} & = & 4\frac{4}{6} \\ 1\frac{1}{6} & = & 1\frac{1}{6} \\ \hline & & 5\frac{5}{6} \end{array}$$

e.
$$\begin{array}{ccccccc} \frac{7}{18} & = & \frac{7}{2 \times 3 \times 3} & = & \frac{3 \times 7}{3 \times (2 \times 3 \times 3)} & = & \frac{21}{54} \\ \frac{4}{27} & = & \frac{4}{3 \times 3 \times 3} & = & \frac{2 \times 4}{2 \times (3 \times 3 \times 3)} & = & \frac{8}{54} \\ \hline & & & & & & \frac{29}{54} \end{array}$$

Figure 20-3. Some sample additions.

Problems

3. Change each of the following pairs of fractions to equivalent pairs having the lowest common denominator.

   a. $\frac{2}{3}, \frac{3}{4}$ b. $\frac{14}{15}, \frac{5}{18}$ c. $\frac{1}{13}, \frac{1}{11}$

4. Add: $7\frac{1}{5} + 9\frac{3}{4}$ using the vertical form model of Figure 20-3d.

5. Add: $\frac{7}{9} + \frac{55}{6}$ using the horizontal form model of Figure 20-3a.

The Properties of Addition of Rational Numbers

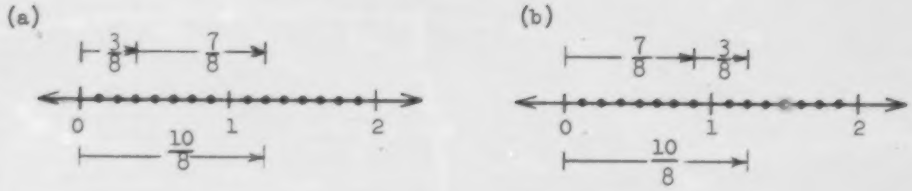
The definition $\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}$ gives us the means to show that addition of rational numbers is commutative and associative. Once this is done we can apply these properties repeatedly in any computation. The effect of this is again a sort of "do it whichever way you please" principle which indicates that so long as only addition is involved it makes no difference in what order you choose to add pairs of numbers or how you choose to group the numbers for purposes of addition. Hence, the two additions indicated in Figure 20-4 should give the same result, since exactly the same numbers are involved, and the second is clearly easier to handle than the first.

a. $\frac{5}{9} + \frac{3}{4} + \frac{5}{6} + \frac{2}{9} + \frac{1}{4} + \frac{1}{6} = \frac{20}{36} + \frac{27}{36} + \frac{30}{36} + \frac{8}{36} + \frac{9}{36} + \frac{6}{36} = \frac{100}{36} = 2\frac{28}{36} = 2\frac{7}{9}$

b. $(\frac{3}{4} + \frac{1}{4}) + (\frac{5}{6} + \frac{1}{6}) + (\frac{5}{9} + \frac{2}{9}) = \frac{4}{4} + \frac{6}{6} + \frac{7}{9} = (1 + 1) + \frac{7}{9} = 2\frac{7}{9}$

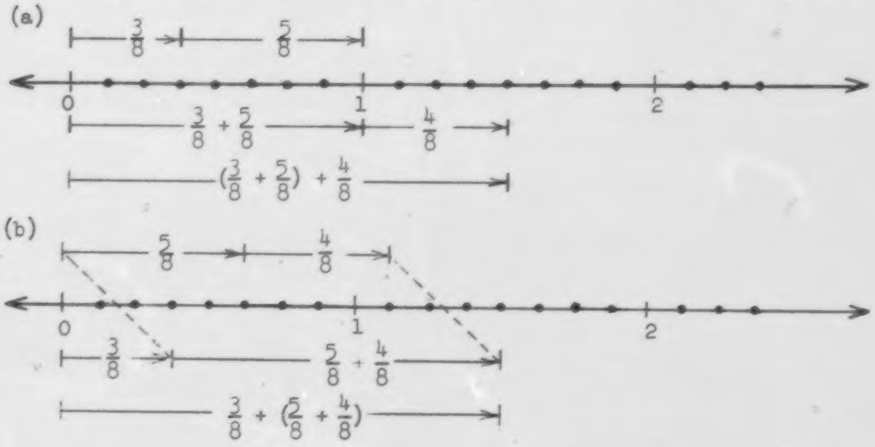
Figure 20-4. Application of a "do it whichever way you please" principle to an addition problem.

We can convince ourselves of the validity of the commutative and associative properties of addition of rational numbers either via the number line, as illustrated in Figures 20-5 and 20-6, or by working a number of specific examples; e.g., $\frac{2}{3} + \frac{4}{3} = \frac{2 + 4}{3} = \frac{6}{3}$ while $\frac{4}{3} + \frac{2}{3} = \frac{4 + 2}{3} = \frac{6}{3}$, so $\frac{2}{3} + \frac{4}{3} = \frac{4}{3} + \frac{2}{3}$.



$$\frac{3}{8} + \frac{7}{8} = \frac{7}{8} + \frac{3}{8}$$

Figure 20-5. An example of the commutative property of addition.



$$\left(\frac{3}{8} + \frac{5}{8}\right) + \frac{4}{8} = \frac{3}{8} + \left(\frac{5}{8} + \frac{4}{8}\right)$$

Figure 20-6. An example of the associative property of addition.

To get a general "proof" that $\frac{a}{b} + \frac{c}{b} = \frac{c}{b} + \frac{a}{b}$, that is, that addition of rational numbers is commutative, we proceed as follows:

> By the definition $\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}$. Since $\underline{a}$ and $\underline{c}$ are whole numbers and since addition of whole numbers is commutative, $a + c$ can be replaced by $c + a$ so our last expression becomes $\frac{c + a}{b}$. Using our definition in reverse this becomes $\frac{c}{b} + \frac{a}{b}$. Written on one line this chain of events is:
>
> $$\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b} = \frac{c + a}{b} = \frac{c}{b} + \frac{a}{b}. \quad \text{Hence,} \quad \frac{a}{b} + \frac{c}{b} = \frac{c}{b} + \frac{a}{b}.$$

Observe that our demonstration uses only facts already known about whole numbers along with the definition of addition of rational numbers. Proofs of other properties are left to the reader in the problems that follow.

Problems

6. A proof that addition of rational numbers is associative is given below with some blanks in it. Fill in the blanks.

$$\left(\frac{a}{b} + \frac{c}{b}\right) + \frac{d}{b} = \frac{a + c}{b} + \frac{\underline{\quad}}{b} = \frac{(\quad\quad) + d}{b} = \frac{a + (c + d)}{b}$$

$$= \frac{a}{b} + \frac{(\quad\quad)}{b} = \frac{a}{b} + \left(\frac{c}{b} + \frac{d}{b}\right)$$

7. At what point in the proof above is the fact that addition of whole numbers is associative used?

8. In what way does the definition of addition, $\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}$, and the fact that the set of whole numbers is "closed" under addition assure us that addition of rational numbers is "closed"? (All we need to be sure of is that addition of two numbers of the form $\frac{a}{b}$, $\underline{a}$ a whole number and $\underline{b}$ a counting number, gives a fraction which is also a whole number over a counting number.)

9. Zero has the property that for whole numbers $a + 0 = a$. An analogous property for rational numbers would appear as $\frac{a}{b} + 0 = \frac{a}{b}$. Remembering that 0 can be represented as $\frac{0}{b}$, and using the definitions of addition of rational numbers, show that this addition property of 0 holds for addition of rational numbers.

Subtraction of Rational Numbers

Turning to subtraction of rational numbers we see that if the denominators are the same, a definition similar to that for addition suffices.

Given two fractions $\frac{a}{b}$ and $\frac{c}{b}$, where $\frac{a}{b} > \frac{c}{b}$,

$$\frac{a}{b} - \frac{c}{b} = \frac{a - c}{b} .$$

You will recall from our discussion of "order" that the specification $\frac{a}{b} > \frac{c}{b}$ assures us that for whole numbers $\underline{a}$ and $\underline{c}$ $a > c$, so that the subtraction $a - c$ can be done.

An alternative way of defining subtraction would be to follow the model given by the definition of subtraction of whole numbers in Chapter 6, namely, "$a - b$ is the whole number $\underline{n}$ for which $n + b = a$." Translating this using rational numbers, a definition of subtraction would appear as follows:

$\frac{a}{b} - \frac{c}{b}$ is that rational number $\frac{n}{b}$ for which $\frac{n}{b} + \frac{c}{b} = \frac{a}{b}$.

In Chapter 6 this was described as the "process of finding an unknown addend." We also recognize this as the specification of how we "check" the supposed answer to any subtraction problem.

The definition " $\frac{a}{b} - \frac{c}{b} = \frac{a - c}{b}$ " gives us an immediate way to find an answer while the second definition, " $\frac{a}{b} - \frac{c}{b}$ is that rational number $\frac{n}{b}$ for which $\frac{n}{b} + \frac{c}{b} = \frac{a}{b}$," is closer to what was done for subtraction of whole numbers. We can show that we can use either definition according to our convenience by showing that the result given by the first definition meets the specification set down by the second definition. This is done below:

> Using the first definition we get $\frac{a}{b} - \frac{c}{b} = \frac{a - c}{b} = \frac{n}{b}$.
> Putting this in the second definition,
>
> $$\frac{n}{b} + \frac{c}{b} = \frac{a - c}{b} + \frac{c}{b} = \frac{(a - c) + c}{b} .$$
>
> But $(a - c) + c = a$ by the inverse property of addition and subtraction. Hence, $\frac{(a - c) + c}{b} = \frac{a}{b}$ as required by the second definition.

As to models for subtraction, we can refer back to Figure 20-2 and think in each case of "taking apart" or "undoing" each of the addition problems illustrated there. Such a process would result in a subtraction problem associated with each such addition problem as shown in the listing below:

Addition problem from Figure 20-2.	Associated Subtraction Problem.
a. $\frac{2}{5} + \frac{1}{5} = \frac{3}{5}$	a. $\frac{3}{5} - \frac{1}{5} = \frac{2}{5}$
b. $\frac{5}{8} + \frac{4}{8} = \frac{9}{8}$	b. $\frac{9}{8} - \frac{4}{8} = \frac{5}{8}$
c. $\frac{6}{10} + \frac{5}{10} = \frac{11}{10}$	c. $\frac{11}{10} - \frac{5}{10} = \frac{6}{10}$

As with addition, the subtraction of rational numbers that are named by fractions with different denominators can be handled by changing them to fractions with the same denominator. Again one finds a common denominator, and preferably a least common denominator, then uses the principle $\frac{a}{b} = \frac{k \times a}{k \times b}$ to get the required fractions with this common denominator.

To carry through the program of verifying properties begun with respect to addition earlier in this chapter, let us verify that properties analagous to those we listed for the subtraction of whole numbers in

Chapters 6 and 7 hold for subtraction of rational numbers. These properties are listed in Figure 20-7, along with their analagous statements for rational numbers and some examples using rational numbers. We assume in this listing that the usual cautions about zero denominators and making sure that subtraction is possible have been made and observed.

Name of subtraction property	Property stated for whole numbers	Property stated for rational numbers
Inverse Properties	$(a - b) + b = a$	$(\frac{a}{b} - \frac{c}{b}) + \frac{c}{b} = \frac{a}{b}$, e.g., $(\frac{3}{8} - \frac{2}{8}) + \frac{2}{8} = \frac{3}{8}$
	$(a + b) - b = a$	$(\frac{a}{b} + \frac{c}{b}) - \frac{c}{b} = \frac{a}{b}$, e.g., $(\frac{3}{8} + \frac{2}{8}) - \frac{2}{8} = \frac{3}{8}$
Properties of Zero	$a - a = 0$	$\frac{a}{b} - \frac{a}{b} = 0$, e.g., $\frac{5}{8} - \frac{5}{8} = \frac{5 - 5}{8} = \frac{0}{8} = 0$
	$a - 0 = a$	$\frac{a}{b} - 0 = \frac{a}{b}$, e.g., $\frac{5}{8} - \frac{0}{8} = \frac{5 - 0}{8} = \frac{5}{8}$
Regrouping Property	$(a + b) - (c + d)$ $= (a - c) + (b - d)$	$(\frac{a}{b} + \frac{c}{b}) - (\frac{d}{b} + \frac{e}{b}) = (\frac{a}{b} - \frac{d}{b}) + (\frac{c}{b} - \frac{e}{b})$ e.g., $3\frac{6}{8} - 1\frac{5}{8} = (3 + \frac{6}{8}) - (1 + \frac{5}{8})$ $= (3 - 1) + (\frac{6}{8} - \frac{5}{8})$ $= 2 + \frac{1}{8} = 2\frac{1}{8}$

Figure 20-7. Properties of subtraction.

You will recall from our discussion in connection with the subtraction of whole numbers that the inverse properties state that subtracting a number and adding the same number are inverse operations. We sometimes describe this in terms of "doing and undoing." It is easy to verify, say on the number line, that the corresponding notions for rational numbers are valid as well.

The properties of zero are easily verified.

The regrouping property was used in Chapter 7 both to show the relation of our place value system of numeration to the ordinary computational processes we use for subtraction of whole numbers and to justify "borrowing" or "regrouping" in such computations. Similarly, this property gives us the means to justify computational procedures in working with rational numbers. This is best explained by examples, one of which appears in the listing of the property in Figure 20-7. Two more such examples are worked out in some detail in Figure 20-8.

$$\begin{aligned} \text{a.}\quad 105\tfrac{4}{5} - 21\tfrac{2}{10} &= (105 + \tfrac{4}{5}) - (21 + \tfrac{2}{10}) \\ &= (105 - 21) + (\tfrac{4}{5} - \tfrac{2}{10}) \\ &= 34 + (\tfrac{8 - 2}{10}) \\ &= 84 + \tfrac{6}{10} \\ &= 84\tfrac{6}{10} \text{ or } 84\tfrac{3}{5} \text{ (since } \tfrac{6}{10} = \tfrac{3}{5}) \end{aligned}$$

$$\begin{array}{lrcccccc} \text{b.} & 27\frac{3}{14} & = & 26 + 1 + \frac{3}{14} & = & 26 + \frac{14}{14} + \frac{3}{14} & = & 26 + \frac{17}{14} \\ & \underline{-\ 14\frac{4}{14}} & = & \underline{14 + \quad \frac{4}{14}} & = & \underline{14 + \quad \frac{4}{14}} & = & \underline{14 + \frac{4}{14}} \\ & & & & & & & 12 + \frac{13}{14} \text{ or } 12\frac{13}{14} \end{array}$$

Figure 20-8.

Problem

10. Following one of the examples given, show in some detail how one would use the regrouping property in doing the subtraction $14\frac{1}{3} - 7\frac{5}{6}$.

Summary

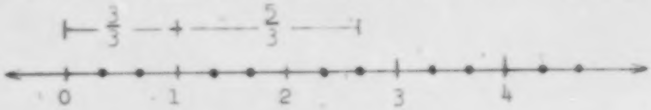
In Chapters 18 and 19 rational numbers, various physical models, equivalence of fractions, ordering of rational numbers, and such notions as raising to higher terms, reducing to lower terms, and lowest common denominator were discussed. The focus there was on individual numbers or relations between such numbers. The present chapter began to focus on binary operations using rational numbers, and in particular, the operations of addition and subtraction. We supplied definitions of these operations for the case of fractions with the same denominator and observed that any two fractions could be made to fit these definitions via equivalent fractions and common denominators. Since such a definition as $\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}$ gets us back immediately to ordinary addition of whole numbers, it is easily verified that our "new" operations using our "new" numbers have essentially the same properties as the addition and subtraction that we have become accustomed to.

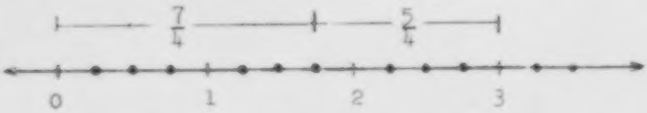
"Mixed numerals" and various computational devices were briefly discussed.

Finally, it should be observed that the various conceptual models for subtraction that were discussed in Chapter 6, such as "take away" versus "how much more," are also applicable here, even though they were not discussed explicitly.

Exercises - Chapter 20

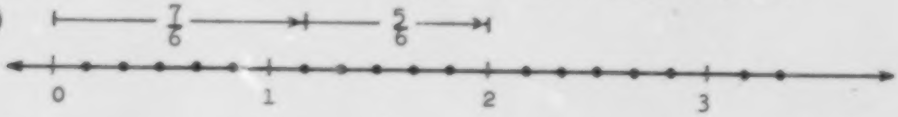
1. Change $\frac{5}{6}$ and $\frac{2}{5}$ to fractions with common denominators so that the denominator is:
   a. a number between 25 and 35;
   b. a number between 40 and 90.
2. Find equivalent numerals with common denominators for $\frac{1}{8}$ and $\frac{3}{7}$, so the denominator is:
   a. a number between 100 and 150;
   b. a number less than 50.
3. Find fractions with lowest common denominators for each set of fractions below:
   a. $\frac{3}{4}$, $\frac{1}{5}$ and $\frac{5}{6}$ b. $\frac{5}{12}$, $\frac{3}{16}$ and $\frac{2}{3}$
4. Draw number line diagrams to show each of these sums:
   a. $\frac{5}{3} + \frac{4}{3}$
   b. $\frac{2}{4} + \frac{7}{4}$
   c. $\frac{3}{2} + \frac{4}{2}$
5. Draw a diagram using unit regions for each of the sums of Exercise 4.
6. What sums are pictured in each of the following diagrams?

(a)


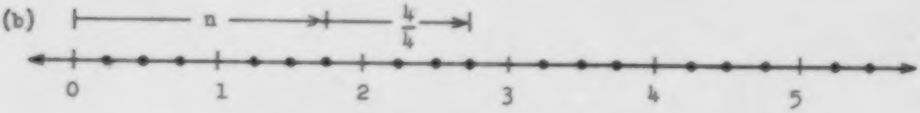
(b)


7. What mathematical sentences are pictured in the diagrams below?

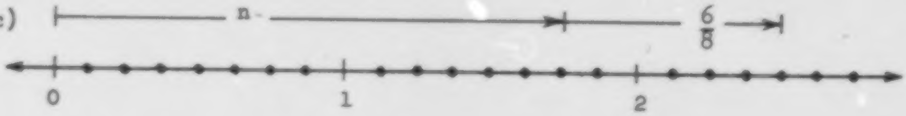
(a)



(b)



(c)



8. a. What is n, if $(\frac{8}{2} - \frac{3}{2}) - \frac{1}{2} = n$?

b. What is n, if $\frac{8}{2} - (\frac{3}{2} - \frac{1}{2}) = n$?

c. Does $(\frac{8}{2} - \frac{3}{2}) - \frac{1}{2} = \frac{8}{2} - (\frac{3}{2} - \frac{1}{2})$?

d. Does the associative property hold for subtraction?

9. Do each of the following in the form indicated by the way the problem is written (vertical or horizontal), then express the answer as a fraction in lowest terms or as a mixed numeral as appropriate.

a. $\frac{7}{8} + \frac{2}{3}$

b. $\begin{array}{r} 7\frac{1}{8} \\ + 4\frac{7}{12} \\ \hline \end{array}$

c. $7\frac{1}{8} + 4\frac{7}{12}$

d. $3\frac{3}{4} + 7\frac{1}{8} - 4\frac{2}{5}$

e. $\begin{array}{r} 6\frac{1}{8} \\ - 3\frac{2}{3} \\ \hline \end{array}$

f. $\frac{7}{8} - \frac{5}{6}$

Solutions for Problems

1. a. 5 b. 1, 6 c. 17, 17

2. $\frac{1}{8} + \frac{3}{8} = \frac{1 + 3}{8} = \frac{4}{8} = \frac{1}{2}$; $\frac{2}{16} + \frac{6}{16} = \frac{2 + 6}{16} = \frac{8}{16} = \frac{1}{2}$

3. a. $\frac{8}{12}$, $\frac{9}{12}$ (since the l.c.m. of 3 and 4 is 12)

   b. $\frac{14}{15} = \frac{14}{3 \times 5} = \frac{14 \times (2 \times 3)}{3 \times 5 \times (2 \times 3)} = \frac{84}{90}$

   $\frac{5}{18} = \frac{5}{2 \times 3 \times 3} = \frac{5 \times 5}{(2 \times 3 \times 3) \times 5} = \frac{25}{90}$

   c. $\frac{1}{13} = \frac{1 \times 11}{13 \times 11} = \frac{11}{143}$

   $\frac{1}{11} = \frac{1 \times 13}{11 \times 13} = \frac{13}{143}$

4. $$\begin{array}{rl} & 7\frac{1}{5} = 7\frac{4}{20} \\ + & 9\frac{3}{4} = 9\frac{15}{20} \\ \hline & \quad\;\; 16\frac{19}{20} \end{array}$$

5. $\frac{7}{9} + \frac{55}{6} = \frac{14}{18} + \frac{165}{18} = \frac{179}{18} = \frac{162 + 17}{18} = \frac{162}{18} + \frac{17}{18} = 9 + \frac{17}{18} = 9\frac{17}{18}$

6. d; a + c; c + d

7. In going from the numerator $(a + c) + d$ to the numerator $a + (c + d)$

8. Closure under addition of whole numbers assures us that the number $a + c$ is a whole number. Since $\underline{b}$ is a counting number, $\frac{a + c}{b}$ is surely a rational number.

9. $\frac{a}{b} + 0 = \frac{a}{b} + \frac{0}{b} = \frac{a + 0}{b} = \frac{a}{b}$

10. Either: $$\begin{aligned} 14\frac{1}{3} - 7\frac{5}{6} = (13 + 1 + \tfrac{1}{3}) - (7 + \tfrac{5}{6}) &= (13 + \tfrac{4}{3}) - (7 + \tfrac{5}{6}) \\ &= (13 - 7) + (\tfrac{4}{3} - \tfrac{5}{6}) \\ &= (13 - 7) + (\tfrac{8}{6} - \tfrac{5}{6}) \\ &= (13 - 7) + (\tfrac{8 - 5}{6}) \\ &= 6 + \tfrac{3}{6} \\ &= 6\tfrac{3}{6} \text{ or } 6\tfrac{1}{2} \end{aligned}$$

or: $$\begin{array}{rcccccc} 14\frac{1}{3} & = & 13 + \frac{3}{3} + \frac{1}{3} & = & 13 + \frac{4}{3} & = & 13 + \frac{8}{6} \\ -\;7\frac{5}{6} & = & 7 + \frac{5}{6} & = & 7 + \frac{5}{6} & = & 7 + \frac{5}{6} \\ \hline & & & & & & 6 + \frac{3}{6} \end{array}$$ or $6\frac{3}{6}$ or $6\frac{1}{2}$

Chapter 21

MULTIPLICATION OF RATIONAL NUMBERS

Introduction

In the last chapter we defined addition for rational numbers in a way that used only operations on whole numbers and we showed that this addition has such properties as closure, commutativity and associativity that are characteristic of addition of whole numbers. Similarly, we defined subtraction of rational numbers in terms of operations on whole numbers and showed that the expected properties are applicable. In each case the binary operation involves taking two numbers and associating with them a third number, say the "sum," or "difference." Our tasks for multiplication of rational numbers are clearly of the same sort. With each pair of rational numbers we want to associate a number called the "product." We want ways of doing this that involve only previously learned operations and concepts. And we would like the properties of such a multiplication to be pretty much the same as those of the now familiar multiplication of whole numbers. Furthermore, in order to be consistent with our efforts so far, we want to find physical models which justify and give content to the procedures we develop.

Now multiplication of rational numbers is really quite a different thing from multiplication of whole numbers and the models for multiplication of whole numbers do us very little good. "Repeated addition," for example, is all very well for $6 \times 3 = 3 + 3 + 3 + 3 + 3 + 3$ and can even serve for $6 \times \frac{1}{3} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{6}{3}$. But it won't do at all for $\frac{1}{3} \times 6$ because it is surely nonsense to speak of "6 added to itself $\frac{1}{3}$ times." Even if we assume some sort of commutativity to make $\frac{1}{3} \times 6 = 6 \times \frac{1}{3}$ to get us out of this pickle, we are in a hopeless bind if we then ask about $\frac{1}{6} \times \frac{1}{3}$. Nor is the "$n \times m$ array" model of much use, for while it is easy to speak of 6×3 as meaning the number of things in an array with six rows of 3 things each, it is difficult to find a sensible statement to describe a $\frac{1}{6}$ by $\frac{1}{3}$ "array" of things. Teachers must surely be aware of such difficulties themselves in order to help youngsters understand this operation which has the same name as and similar properties to the corresponding whole number operation but which is quite different conceptually.

Models for Multiplication

The principal models available to us as a basis for defining a "multiplication" for rational numbers are closely tied to our intuitive notions of the meaning of the word "of" in such statements as "$\frac{1}{2}$ of them;" "$\frac{1}{3}$ of the $\frac{1}{2}$ gallon of milk," "find what part of a mile $\frac{2}{3}$ of $\frac{1}{8}$ of a mile is;" and so on. Let us then show some models using an "of" operation, then define multiplication by identifying it with this operation. Admittedly this sounds like a rather slippery procedure, but if we are to provide physical models in terms of concepts now available to us, it is the only procedure we have. The alternative is to give a purely abstract definition using already known operations. But in this book we have avoided whenever possible the making of abstract definitions unsupported by models, as we probably should in teaching youngsters.

Suppose we want a model for "$\frac{1}{2}$ of $\frac{1}{3}$" in terms, say, of a unit region. The first step, familiar to us by now, is to represent $\frac{1}{3}$ as one of three congruent parts of the unit region as shown in Figure 21-1a. Let us now regard the region representing $\frac{1}{3}$ as, in a sense, a unit region itself and represent $\frac{1}{2}$ of it by marking 1 of two congruent regions, as shown in Figure 21-1b. The resulting region, representing $\frac{1}{2}$ of $\frac{1}{3}$, is 1 of 6 congruent parts of the original unit region, as can be seen from Figure 21-1b. Hence $\frac{1}{2}$ of $\frac{1}{3} = \frac{1}{6}$.

Similarly, Figure 21-2 shows $\frac{2}{3}$ of $\frac{1}{5}$, which is clearly shown to be 2 of 15 congruent parts of the unit region we started with. Hence, $\frac{2}{3}$ of $\frac{1}{5} = \frac{2}{15}$.

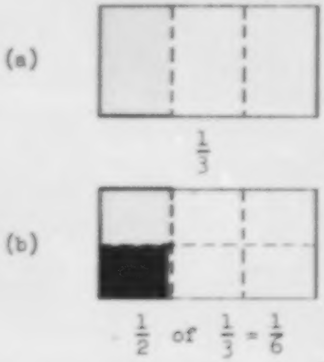


Figure 21-1. Model for $\frac{1}{2}$ of $\frac{1}{3}$.

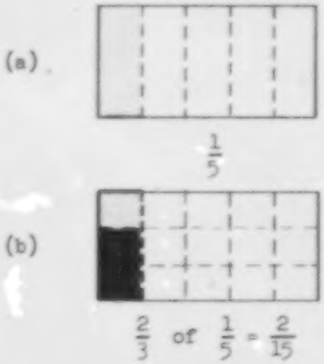


Figure 21-2. Model for $\frac{2}{3}$ of $\frac{1}{5}$.

Likewise, Figure 21-3 first represents $\frac{5}{6}$ in terms of a unit region, then shows $\frac{4}{5}$ of the region representing $\frac{5}{6}$; with a resulting region that is $\frac{20}{30}$ of the unit region.

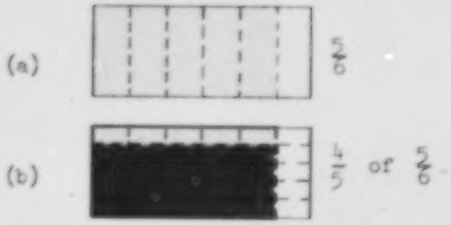


Figure 21-3. Model for $\frac{4}{5}$ of $\frac{5}{6} = \frac{20}{30}$.

Observe that we work such problems just by construction of congruent regions using directions given in terms of counting numbers; then get our answer by counting the number of resulting congruent regions in the unit region and how many of these are marked. That is, we need use only quite fundamental notions. We could get closer to "multiplication" by observing that the final result of, say, $\frac{4}{5}$ of $\frac{5}{6}$ (Figure 21-3), shows 4 rows, each containing 5 congruent parts, which is a sort of 4 by 5 array. Furthermore, there are $5 \times 6 = 30$ such congruent parts in the base unit because there are 5 rows each containing 6 such parts; or a 5×6 array. Hence $\frac{4}{5}$ of $\frac{5}{6}$ will be 4×5 parts out of 5×6 parts $= \frac{4 \times 5}{5 \times 6} = \frac{20}{30}$. Similarly, our model for $\frac{2}{3}$ of $\frac{1}{5}$ (Figure 21-2) ultimately results in a 3×5 array of congruent parts comprising the base unit and a 2×1 array of such parts comprising the end result. That is, $\frac{2}{3}$ of $\frac{1}{5} = \frac{2 \times 1}{3 \times 5} = \frac{2}{15}$.

Problems *

1. Draw a model starting from unit regions for each of the following, as is done in Figures 21-1, 21-2 and 21-3.
   a. $\frac{1}{2}$ of $\frac{3}{4}$
   b. $\frac{2}{4}$ of $\frac{7}{8}$
   c. $\frac{2}{3}$ of $\frac{3}{4}$
   d. $\frac{3}{4}$ of $\frac{2}{3}$
2. Observe from c. and d. above that while $\frac{2}{3}$ of $\frac{3}{4}$ and $\frac{3}{4}$ of $\frac{2}{3}$ give different models, the number that results is the same. Do you think the "of" operation is commutative?
3. Without actually drawing a model, answer the following with reference to " $\frac{2}{3}$ of $\frac{7}{8}$."
   a. Into how many congruent parts will the unit region ultimately be divided in representing $\frac{7}{8}$ and then representing $\frac{2}{3}$ of $\frac{7}{8}$?
   b. Of these, how many will be marked to show $\frac{2}{3}$ of $\frac{7}{8}$?

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* Solutions to problems in this chapter are on page

4. Fill in the blanks: In representing $\frac{5}{7}$ of $\frac{8}{9}$ we first divide the unit into ____ congruent parts using vertical segments, and we mark ____ of these. We then divide the resulting representation of $\frac{8}{9}$ into congruent parts using horizontal segments. This results in a ____ by ____ array of congruent parts covering the unit region, of which a ____ by ____ array is marked to represent $\frac{5}{7}$ of $\frac{8}{9}$. Hence $\frac{5}{7}$ of $\frac{8}{9} = \frac{5 \times \underline{\quad}}{\underline{\quad} \times \underline{\quad}} = \frac{40}{63}$.

As usual, we can also use the number line as a model of, say, $\frac{1}{3}$ of $\frac{4}{5}$, though this model is much harder to follow. As illustrated in Figure 21-4, we first mark $\frac{4}{5}$ on the number line in the usual way. Then, considering the segment $\overline{AB}$, with $m(\overline{AB}) = \frac{4}{5}$, we mark the point C that is $\frac{1}{3}$ of the way along $\overline{AB}$. In terms of our original unit, $m(\overline{AC}) = \frac{1}{3}$ of $\frac{4}{5}$ and we can surely use this to mark a point corresponding to $\frac{1}{3}$ of $\frac{4}{5}$ on the number line. However, our procedure so far has not given us a fraction to name this point, and this is the difficulty with the number line model. As shown in Figure 21-4b, we must resort to equivalent fractions and observe that had we divided the segment into 15 parts and taken 12 of them, the point named $\frac{12}{15}$ would be the same point as that named $\frac{4}{5}$. Now taking a point $\frac{1}{3}$ of the way along on this segment, we arrive at $\frac{4}{15}$. Hence, $\frac{1}{3}$ of $\frac{4}{5} = \frac{4}{15}$. Essentially we need to rig things via equivalent fractions so that the unit segment is marked in such a way that all our work will come out exactly on one of the original division points from the unit segment. This can always be done by using the product of the two denominators as the number of congruent parts in the original unit segment; you may want to work through enough examples to see why this is so. Figure 21-5 provides two other examples. Clearly, however, the unit region notion provides a model easier to manipulate than does the number line. On the other hand, the number line is a closer approximation to such situations as those in Problem 6 below.

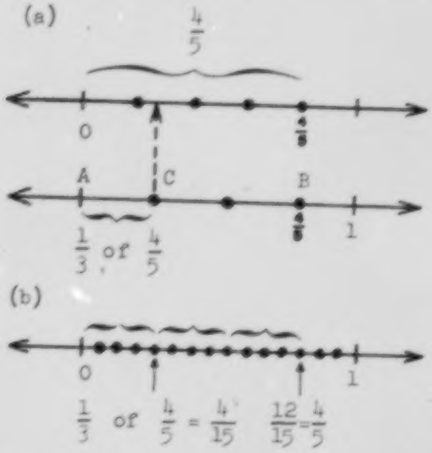
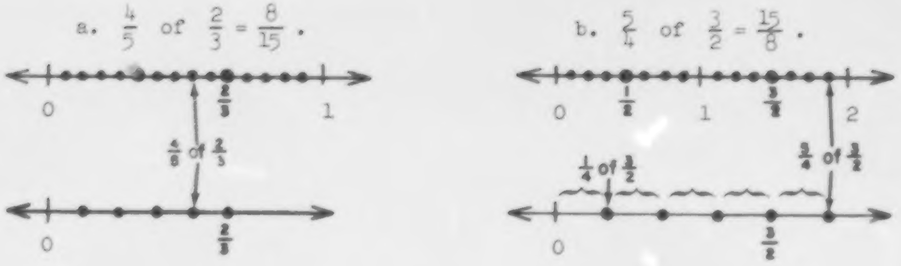

(a)
$\frac{4}{5}$
0
$\frac{4}{5}$
A
C
B
$\frac{4}{5}$
1
$\frac{1}{3}$ of $\frac{4}{5}$
(b)
0
1
$\frac{1}{3}$ of $\frac{4}{5} = \frac{4}{15}$
$\frac{12}{15} = \frac{4}{5}$

Figure 21-4. Number line model for $\frac{1}{3}$ of $\frac{4}{5} = \frac{4}{15}$.



a. $\frac{4}{5}$ of $\frac{2}{3} = \frac{8}{15}$.

b. $\frac{5}{4}$ of $\frac{3}{2} = \frac{15}{8}$.

0
$\frac{2}{3}$
1
$\frac{4}{5}$ of $\frac{2}{3}$
0
$\frac{2}{3}$
0
$\frac{1}{2}$
1
$\frac{3}{2}$
2
$\frac{1}{4}$ of $\frac{3}{2}$
$\frac{5}{4}$ of $\frac{3}{2}$
0
$\frac{3}{2}$

Figure 21-5. Number line models showing $\frac{4}{5}$ of $\frac{2}{3} = \frac{8}{15}$ and $\frac{5}{4}$ of $\frac{3}{2} = \frac{15}{8}$.

Problems

5. In the diagram below, name segments whose measures are:

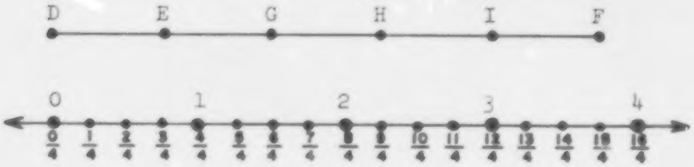
   a. $\frac{3}{4}$

   b. 5 of $\frac{3}{4}$

   c. $\frac{15}{4}$

   d. $\frac{3}{2}$

   e. $\frac{1}{2}$ of $\frac{3}{2}$



D
E
G
H
I
F
0
1
2
3
4
$\frac{0}{4}$
$\frac{1}{4}$
$\frac{2}{4}$
$\frac{3}{4}$
$\frac{4}{4}$
$\frac{5}{4}$
$\frac{6}{4}$
$\frac{7}{4}$
$\frac{8}{4}$
$\frac{9}{4}$
$\frac{10}{4}$
$\frac{11}{4}$
$\frac{12}{4}$
$\frac{13}{4}$
$\frac{14}{4}$
$\frac{15}{4}$
$\frac{16}{4}$

6. Show on the number line $\frac{1}{2} \times \frac{1}{3}$.

7. Picture the following problems on the number line and give "of" problems using fractions that fit the situation.

   a. Sue used $\frac{3}{4}$ of a piece of toweling to make a place mat. If the piece was $\frac{2}{3}$ yard long, how long a piece did she use for the place mat?

   b. The Scouts hiked from the school to a camp $3\frac{1}{2}$ miles away. They stopped to rest when they had gone $\frac{1}{3}$ of the way. How far had they walked when they stopped to rest?

The Definition of Multiplication for Rational Numbers

In order to define multiplication we identify the "of" operation with multiplication; that is, $\frac{1}{2} \times \frac{1}{3} = \frac{1}{2}$ of $\frac{1}{3}$; $\frac{2}{3} \times \frac{4}{5} = \frac{2}{3}$ of $\frac{4}{5}$; etc. The "=" sign means, as usual, that the same number will result. Next, we tabulate in Figure 21-6 the results of all the examples we have considered so far in this chapter and observe that in every case the numerator of the result is the product of the numerators of the fractions, while the denominator of the result is the product of the denominators of the fractions. (You should take a moment and verify this.)

$$\frac{1}{2} \times \frac{1}{3} = \frac{1}{2} \text{ of } \frac{1}{3} = \frac{1}{6}$$

$$\frac{2}{3} \times \frac{1}{5} = \frac{2}{3} \text{ of } \frac{1}{5} = \frac{2}{15}$$

$$\frac{4}{5} \times \frac{5}{6} = \frac{4}{5} \text{ of } \frac{5}{6} = \frac{20}{30}$$

$$\frac{2}{4} \times \frac{7}{8} = \frac{2}{4} \text{ of } \frac{7}{8} = \frac{14}{32}$$

$$\frac{2}{3} \times \frac{3}{4} = \frac{2}{3} \text{ of } \frac{3}{4} = \frac{6}{12}$$

$$\frac{3}{4} \times \frac{2}{3} = \frac{3}{4} \text{ of } \frac{2}{3} = \frac{6}{12}$$

$$\frac{2}{3} \times \frac{7}{8} = \frac{2}{3} \text{ of } \frac{7}{8} = \frac{14}{24}$$

$$\frac{5}{7} \times \frac{8}{9} = \frac{5}{7} \text{ of } \frac{8}{9} = \frac{40}{63}$$

$$\frac{1}{3} \times \frac{4}{5} = \frac{1}{3} \text{ of } \frac{4}{5} = \frac{4}{15}$$

Figure 21-6. The "×" and "of" operations.

Hence we are led to the definition:

Given two fractions $\frac{a}{b}$ and $\frac{c}{d}$, $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$.

This definition gives a computational procedure that depends only on multiplication of whole numbers. As with whole numbers we will call $\frac{a}{b} \times \frac{c}{d}$ the "product" of the "factors" $\frac{a}{b}$ and $\frac{c}{d}$.

Various refinements of the computational procedure given by the definition, such as "cancelling," will be discussed later in this chapter. For now, let us just observe that whole numbers and mixed numerals can also be handled with this definition by use of fractions equivalent to them, as illustrated below:

$$4 \times \frac{2}{5} = \frac{4}{1} \times \frac{2}{5} = \frac{4 \times 2}{1 \times 5} = \frac{8}{5}$$

$$7 \times 8 = \frac{7}{1} \times \frac{8}{1} = \frac{7 \times 8}{1 \times 1} = \frac{56}{1}$$

$$4\frac{1}{3} \times 2\frac{1}{2} = \frac{13}{3} \times \frac{5}{2} = \frac{13 \times 5}{3 \times 2} = \frac{65}{6}$$

Properties of Multiplication

We should now check to see whether or not multiplication as we have defined it for rational numbers has the properties characteristic of multiplication of whole numbers. Figure 21-7 shows each of the whole number properties displayed along with the analogous rational number property and along with one or more examples. Besides the familiar properties, there is one really new property displayed, the reciprocal property, that holds for rational numbers but is not applicable to whole numbers. It is suggested that the reader satisfy himself that each of the properties should hold by studying the chart and the examples. This is especially urged for the reciprocal property, which is new to us.

Whole Number Properties	Rational Number Properties
Suppose that a, b and c are whole numbers:	Suppose that $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}$ are rational numbers:
Definition: $a \times b$ is the number of elements in an a by b array.	Definition: $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$.

Closure

$a \times b$ is a whole number	$\frac{a}{b} \times \frac{c}{d}$ is a rational number

Commutativity

$a \times b = b \times a$	$\frac{a}{b} \times \frac{c}{d} = \frac{c}{d} \times \frac{a}{b}$

Associativity

$(a \times b) \times c = a \times (b \times c)$	$(\frac{a}{b} \times \frac{c}{d}) \times \frac{e}{f} = \frac{a}{b} \times (\frac{c}{d} \times \frac{e}{f})$

1 is an Identity for Multiplication.

$a \times 1 = a$	$\frac{a}{b} \times 1 = \frac{a}{b}$

Multiplication is Distributive over Addition.

$a \times (b + c) = (a \times b) + (a \times c)$	$\frac{a}{b} \times (\frac{c}{d} + \frac{e}{f}) = (\frac{a}{b} \times \frac{c}{d}) + (\frac{a}{b} \times \frac{e}{f})$

Multiplication is Distributive over Subtraction.

$a \times (b - c) = (a \times b) - (a \times c)$	$\frac{a}{b} \times (\frac{c}{d} - \frac{e}{f}) = (\frac{a}{b} \times \frac{c}{d}) - (\frac{a}{b} \times \frac{e}{f})$

Multiplication Prcperty of 0

$a \times 0 = 0$	$\frac{a}{b} \times 0 = 0$

Reciprocal Property

(There is no property of whole numbers that corresponds to the reciprocal property.)	If a and b are counting numbers, $\frac{a}{b} \times \frac{b}{a} = 1$ and $a \times \frac{1}{a} = 1$. $\frac{1}{a}$ is called the reciprocal of a. $\frac{b}{a}$ and $\frac{a}{b}$ are reciprocals of each other.

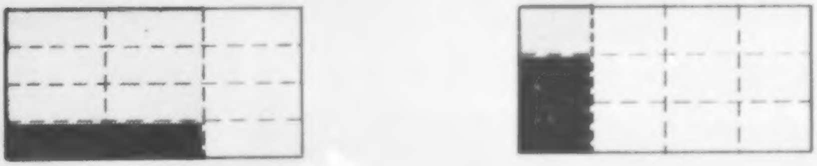
Figure 21-7. Properties of multiplication of whole numbers and of rational numbers.

Proofs of the Properties of Multiplication of Rational Numbers

This rather long section gives demonstrations (proofs) that each of the properties listed in Figure 21-7 holds in general assuming only that we accept the properties of multiplication of whole numbers, the statements of Chapters 19 and 20 about equivalence of fractions and addition of rational numbers, and the definition just given for multiplication of rational numbers. The reader may prefer to convince himself of the validity of the properties by way of examples and skip some or all of the proofs. The proofs are intended mainly to make the point that examples alone, no matter how many are given, do not suffice to prove a property that applies to all numbers, and also to demonstrate how the structure of mathematics is built up step by step on the basis of "first principles" or "primitive notions." In this instance, we have built by now a fairly elaborate structure that has as its base some quite simple notions about "sets" of things.

Closure. "If $\frac{a}{b}$ and $\frac{c}{d}$ are any two rational numbers, $\frac{a}{b} \times \frac{c}{d}$ is a rational number." This can be easily seen by observing that $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$ by definition, where $a \times c$ is a whole number, since whole numbers are closed under multiplication, and $b \times d$ is a counting number, since counting numbers are also closed under multiplication. Therefore $\frac{a \times b}{c \times d}$ is a rational number.

Commutativity. The commutative property may be illustrated for particular numbers using unit regions, as shown in Figure 21-8.



$\frac{1}{4} \times \frac{2}{3} = \frac{2}{12}$ $\frac{2}{3} \times \frac{1}{4} = \frac{2}{12}$

Figure 21-8. Models showing an example of commutative property: $\frac{1}{4} \times \frac{2}{3} = \frac{2}{3} \times \frac{1}{4}$. Although the shapes of these regions are not the same, each of them contains 2 small regions, 12 of which make up the original figures.

It is really easier to check the commutative property directly from the definition. Thus:

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d} \text{ and } \frac{c}{d} \times \frac{a}{b} = \frac{c \times a}{d \times b}.$$

But, since a and c are whole numbers, and b and d are counting numbers, we know $a \times c = c \times a$ and $b \times d = d \times b$, since their multiplication is commutative. So, $\frac{a \times c}{b \times d} = \frac{c \times a}{d \times b}$, and therefore, $\frac{a}{b} \times \frac{c}{d} = \frac{c}{d} \times \frac{a}{b}$.

Associativity. We prove this property directly from our definition of multiplication. For if $\frac{a}{b}$, $\frac{c}{d}$ and $\frac{e}{f}$ are any rational numbers,

$$(\frac{a}{b} \times \frac{c}{d}) \times \frac{e}{f} = \frac{(a \times c)}{(b \times d)} \times \frac{e}{f} = \frac{(a \times c) \times e}{(b \times d) \times f}$$

and $$\frac{a}{b} \times (\frac{c}{d} \times \frac{e}{f}) = \frac{a}{b} \times \frac{(c \times e)}{(d \times f)} = \frac{a \times (c \times e)}{b \times (d \times f)} .$$

But, $a \times (c \times e) = (a \times c) \times e$ and $(b \times d) \times f = b \times (d \times f)$ because the associative property holds for whole numbers and counting numbers. Hence, $\frac{(a \times c) \times e}{(b \times d) \times f} = \frac{a \times (c \times e)}{b \times (d \times f)}$, which means that

$$(\frac{a}{b} \times \frac{c}{d}) \times \frac{e}{f} = \frac{a}{b} \times (\frac{c}{d} \times \frac{e}{f}) .$$

One is an Identity for Multiplication.

$$1 \times \frac{a}{b} = \frac{1}{1} \times \frac{a}{b} = \frac{1 \times a}{1 \times b} = \frac{a}{b}$$

Distributivity

Our unit region model for addition and for multiplication can be used to demonstrate the distributive property. Figure 21-9 shows that $\frac{2}{3} \times (\frac{1}{4} + \frac{2}{4}) = (\frac{2}{3} \times \frac{1}{4}) + (\frac{2}{3} \times \frac{2}{4})$. In Figure 21-9a the regions representing $\frac{1}{4}$ and $\frac{2}{4}$ are "added," then we take $\frac{2}{3}$ of the result. In Figure 21-9b we first find $\frac{2}{3}$ of a $\frac{1}{4}$ unit region, then $\frac{2}{3}$ of a $\frac{2}{4}$ unit region, then "add" these results. The two results are the same. Hence, $\frac{2}{3} \times (\frac{1}{4} + \frac{2}{4}) = (\frac{2}{3} \times \frac{1}{4}) + (\frac{2}{3} \times \frac{2}{4})$

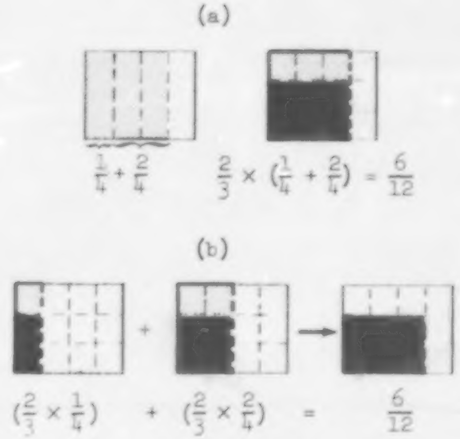

(a)
$\frac{1}{4} + \frac{2}{4}$
$\frac{2}{3} \times (\frac{1}{4} + \frac{2}{4}) = \frac{6}{12}$
(b)
+
$(\frac{2}{3} \times \frac{1}{4})$ + $(\frac{2}{3} \times \frac{2}{4})$ = $\frac{6}{12}$

Figure 21-9. Distributive property demonstrated by "adding" regions.

As before, we can give a general "proof" for the property. This is shown below.

$$\frac{a}{b} \times \left(\frac{c}{d} + \frac{e}{f}\right) = \frac{a}{b} \times \left(\frac{c \times f}{d \times f} + \frac{d \times e}{d \times f}\right) = \frac{a}{b} \times \frac{(c \times f) + (d \times e)}{d \times f}$$

$$= \frac{a \times [(c \times f) + (d \times e)]}{b \times (d \times f)}$$

$$= \frac{[a \times (c \times f)] + [a \times (d \times e)]}{b \times (d \times f)}$$

Also,

$$\left(\frac{a}{b} \times \frac{c}{d}\right) + \left(\frac{a}{b} \times \frac{e}{f}\right) = \frac{a \times c}{b \times d} + \frac{a \times e}{b \times f}$$

$$= \frac{(a \times c) \times f}{(b \times d) \times f} + \frac{(a \times e) \times d}{(b \times f) \times d}$$

(Note that $b \times d \times f$ is used here as a common denominator in the addition.)

$$= \frac{a \times (c \times f)}{b \times (d \times f)} + \frac{a \times (e \times d)}{b \times (f \times d)}$$

$$= \frac{[a \times (c \times f)] + [a \times (d \times e)]}{b \times (d \times f)}$$

Here we have used the associative and commutative properties for multiplication of whole numbers repeatedly. Comparing, we see that the end result for $\frac{a}{b} \times (\frac{c}{d} + \frac{e}{f})$ is the same as for $(\frac{a}{b} \times \frac{c}{d}) + (\frac{a}{b} \times \frac{e}{f})$. Hence,

$$\frac{a}{b} \times \left(\frac{c}{d} + \frac{e}{f}\right) = \left(\frac{a}{b} \times \frac{c}{d}\right) + \left(\frac{a}{b} \times \frac{e}{f}\right)$$

is a true statement; i.e., the distributive property does hold.

An exactly analogous proof shows that multiplication is distributive over subtraction. We will not reproduce this proof here.

The Multiplicative Inverse Property. The rational numbers possess an important property not possessed by either the counting numbers or the whole numbers. The product of $\frac{2}{3}$ and $\frac{3}{2}$ is 1. The product of $\frac{5}{2}$ and $\frac{2}{5}$ is 1. The number $\frac{2}{3}$ is called the multiplicative inverse or the reciprocal of $\frac{3}{2}$. The reciprocal of $\frac{5}{2}$ is $\frac{2}{5}$. The reciprocal of a number is the number by which it must be multiplied to give 1. Every rational number, except 0, has a reciprocal. When a number is named by a fraction, we can easily find its reciprocal by "inverting" the fraction, that is, by interchanging the numerator and the denominator. Thus, the reciprocal of $\frac{5}{8}$ is $\frac{8}{5}$. The reciprocal of $\frac{8}{5}$ is $\frac{5}{8}$. The reciprocal of 2 is $\frac{1}{2}$. (Notice another name for 2 is $\frac{2}{1}$, which inverted is $\frac{1}{2}$.) In general, the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$.

The number 0 has no reciprocal. The product of 0 and every number is 0, hence it is impossible to find a number such that you can multiply it by 0 and get 1.

Proof of the multiplicative inverse property:

a. If $a \neq 0$, $a \times \frac{1}{a} = \frac{a}{1} \times \frac{1}{a} = \frac{a \times 1}{1 \times a} = \frac{a}{a} = 1$.

b. If $a \neq 0$ and $b \neq 0$, $\frac{a}{b} \times \frac{b}{a} = \frac{a \times b}{b \times a} = \frac{a \times b}{a \times b} = 1$, since any number $\frac{h}{h}$ is equivalent to 1, and $\frac{a \times b}{a \times b}$ is certainly such a number.

Problems

8. Write the reciprocals of the numbers.

a. $\frac{11}{1}$ c. $\frac{2}{7}$ e. 3

b. $\frac{1}{5}$ d. $\frac{50}{3}$

9. In the following, the letters represent numbers different from zero. Write the reciprocals.

a. m b. s c. $\frac{1}{c}$ d. $\frac{r}{s}$ e. $\frac{t}{3}$

10. Fill in the blanks to make each statement conform to the multiplicative inverse pattern.

a. $\frac{3}{2} \times$ ____ $= 1$

b. ____ $\times \frac{10}{7} = 1$

Some Computing Considerations

Most of the "shortcuts" used for efficient computation in multiplication of rational numbers just involve looking ahead and using the properties of multiplication and of equivalence. For example, we visualize $\frac{2}{3} \times \frac{3}{2}$ as if it were already written $\frac{2 \times 3}{3 \times 2}$ and see that the answer is 1. It is fairly important to see how such shortcuts are justified so that they don't become a sort of magic way of getting results and so that unjustified analogies are not made. For example, $\frac{2 \times \cancel{3}}{3 \times \cancel{3}} = \frac{2}{3}$ is correct but $\frac{2 + \cancel{3}}{3 + \cancel{3}} = \frac{2}{3}$ is not correct (since $\frac{2 + 3}{3 + 3} = \frac{5}{6}$). Ultimately an important aim of arithmetic instruction is accurate and efficient computation, but a person should probably also be able to exhibit his understanding of the arithmetic processes by being able to justify, on demand, in detail any step in a computation on the basis of properties that have been listed, even if the statement of such properties has been quite informal.

The principle shortcut for multiplication of rational numbers usually goes by the name of "cancelling," though for several reasons this is a misnomer. Such a completed problem might appear as follows:

$$\frac{\overset{7}{\cancel{28}}}{\underset{5}{\cancel{15}}} \times \frac{\overset{3}{9}}{\underset{5}{\cancel{20}}} = \frac{21}{25}.$$

This process can be considered in detail in several ways. Three such detailed displays are given in Figure 21-10b, c, d. Observe that in Figures 21-10b and 21-10c we use the definition first, then factoring and regrouping to get a fraction of the form $\frac{k}{k}$, then the definition of multiplication in reverse to separate $\frac{k}{k}$ from the rest to get 1 times something.

a. $\frac{28}{15} \times \frac{9}{20}$ via "cancelling"

$$\frac{\overset{7}{\cancel{28}}}{\underset{5}{15}} \times \frac{\overset{3}{9}}{\underset{5}{\cancel{20}}} = \frac{21}{25}$$

b. Done via the form $\frac{n}{n} \times \frac{m}{m} \times \frac{x}{y}$

$$\frac{28}{15} \times \frac{9}{20} = \frac{28 \times 9}{15 \times 20} = \frac{4 \times 7 \times 3 \times 3}{3 \times 5 \times 4 \times 5} =$$

$$\frac{4 \times 3 \times 7 \times 3}{4 \times 3 \times 5 \times 5} = \frac{4}{4} \times \frac{3}{3} \times \frac{7 \times 3}{5 \times 5} = 1 \times 1 \times \frac{21}{25} = \frac{21}{25}$$

c. Done via the form $\frac{k}{k} \times \frac{r}{t}$

$$\frac{28}{15} \times \frac{9}{20} = \frac{28 \times 9}{15 \times 20} = \frac{4 \times 7 \times 3 \times 3}{3 \times 5 \times 4 \times 5} = \frac{(4 \times 3) \times (3 \times 7)}{(4 \times 3) \times (4 \times 5)} =$$

$$\left(\frac{4 \times 3}{4 \times 3}\right) \times \left(\frac{3 \times 7}{4 \times 5}\right) = 1 \times \frac{21}{25} = \frac{21}{25}$$

d. "Reduced" by use of the greatest common factor

$$\frac{28}{15} \times \frac{9}{20} = \frac{28 \times 9}{15 \times 20} = \frac{2 \times 2 \times 7 \times 3 \times 3}{3 \times 5 \times 2 \times 2 \times 5} =$$

$$\frac{[(2 \times 2 \times 3) \times (3 \times 7)] \div (2 \times 2 \times 3)}{[(2 \times 2 \times 3) \times (4 \times 5)] \div (2 \times 2 \times 3)} = \frac{3 \times 7}{4 \times 5} = \frac{21}{25}$$

Figure 21-10. Why "cancelling" works.

In Figure 21-10d we use the definition of multiplication to get a single fraction, then "reduce" this fraction by using prime factorization to get the g.c.f. of numerator and denominator and divide both by this g.c.f. A fourth alternative, not shown here, would be to use the definition to express the product as a single fraction, then "reduce" by dividing numerator and denominator by each common factor in turn. This last is probably closest of all to "cancellation" as it is usually done. Observe that even these detailed displays do not exhibit in detail the many applications of the commutative and associative properties in the regrouping process. Observe also that there is no magic in cancelling and that the numerals do not disappear into thin air but rather are replaced by equivalent numerals using the properties we have listed.

Summary

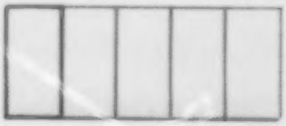
Multiplication of rational numbers is different conceptually and involves some different models than is the case with multiplication of whole numbers. In defining multiplication of rational numbers we rely on an identification of "×" with "of," i.e., $\frac{4}{5}$ of $\frac{2}{7}$ means $\frac{4}{5} \times \frac{2}{7}$. Having defined $\frac{a}{b} \times \frac{c}{d}$ as $\frac{a \times c}{b \times d}$ we verified that all the expected properties of multiplication hold by relying on previously learned properties of whole numbers and rational numbers. Furthermore, one new property was discussed and proved; namely, the multiplicative inverse property, $\frac{a}{b} \times \frac{b}{a} = 1$, where $\frac{a}{b}$ and $\frac{b}{a}$ are said to be reciprocals of each other. This last property will be very useful in defining "division" of rational numbers. The principles that justify the shortcut known as "cancelling" were discussed.

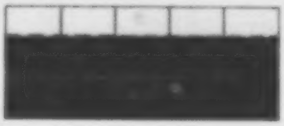
As a final word on the differences between multiplication of rational numbers and multiplication of whole numbers, we should observe that in every multiplication using whole number factors other than zero or one, the product is larger than either factor. This is certainly not the case with rational numbers, and in fact it is often difficult to predict just how large the product will be. Such prediction is even more difficult for division using rational numbers, as will be discussed in the next chapter.

Exercises - Chapter 21

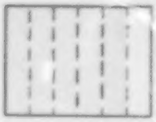
1. For each of the following, write the fraction represented by the region shaded in the first picture, and the multiplication problem and answer which is represented by the shaded part of the second picture. In each case the heavily outlined region in the first picture is the unit region for the problem.

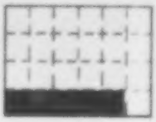
(a)



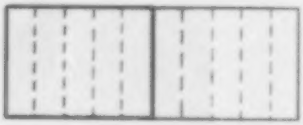


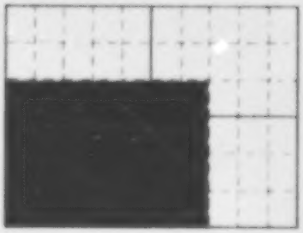
(b)





(c)





2. From the diagram below, name the segments whose measures are:

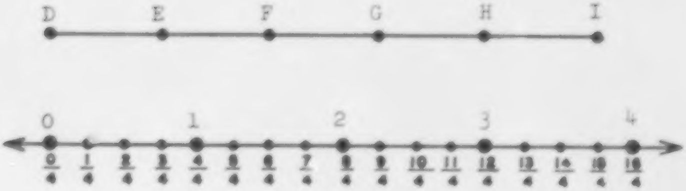
a. $\frac{1}{4} \times 3$

b. $1\frac{1}{2} \times \frac{5}{2}$

c. $\frac{15}{4}$

d. $6 \times \frac{1}{4}$

e. $1 \times \frac{3}{4}$


D
E
F
G
H
I
0
1
2
3
4
0/4
1/4
2/4
3/4
4/4
5/4
6/4
7/4
8/4
9/4
10/4
11/4
12/4
13/4
14/4
15/4
16/4

3. Name the segments whose measures are:

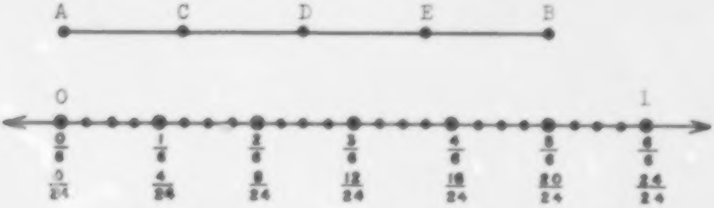
a. $\frac{5}{6}$

b. $\frac{1}{4} \times \frac{5}{6}$

c. $\frac{5}{24}$

d. $\frac{5}{12}$

e. $\frac{1}{2} \times \frac{5}{12}$

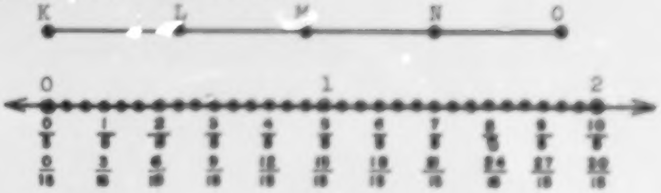

A
C
D
E
B
0
1
0/6
1/6
2/6
3/6
4/6
5/6
6/6
0/24
4/24
8/24
12/24
16/24
20/24
24/24

4. Name the segments whose measures are:

   a. $\frac{7}{5}$

   b. $\frac{4}{3} \times \frac{7}{5}$

   c. $\frac{28}{15}$



5. Use unit regions to show the following:

   a. $\frac{1}{2} \times \frac{2}{3}$ b. $\frac{5}{8} \times \frac{2}{3}$ c. $\frac{2}{3}$ of $1\frac{1}{4}$

6. Use the number line to show:

   a. $\frac{1}{3} \times \frac{1}{2}$ b. $\frac{1}{6} \times \frac{2}{3}$

7. Using Figure 21-9 as a guide, show using unit regions that $\frac{1}{2} \times (\frac{1}{3} + \frac{1}{3}) = (\frac{1}{2} \times \frac{1}{3}) + (\frac{1}{2} \times \frac{1}{3})$.

8. For each of the following use one of the models illustrated in Figure 21-10 or else repeated division by common prime factors to justify in some detail the "cancelling" that is shown. Use each of the models at least once.

   a. $\frac{\overset{1}{\cancel{2}}}{\underset{1}{\cancel{3}}} \times \frac{\overset{5}{\cancel{15}}}{\underset{12}{\cancel{24}}} = \frac{5}{12}$

   c. $\frac{\overset{\overset{1}{\cancel{3}}}{\cancel{12}}}{\underset{\underset{1}{\cancel{7}}}{\cancel{21}}} \times \frac{\overset{1}{\cancel{7}}}{\underset{4}{\cancel{16}}} = \frac{1}{4}$

   b. $\frac{\overset{1}{\cancel{7}}}{\underset{1}{\cancel{3}}} \times \frac{\overset{1}{\cancel{2}}}{\underset{6}{\cancel{12}}} \times \frac{\overset{\overset{1}{\cancel{2}}}{\cancel{6}}}{\underset{\underset{1}{\cancel{7}}}{\cancel{14}}} = \frac{1}{6}$

   d. $\frac{\overset{9}{\cancel{54}}}{\underset{2}{\cancel{12}}} \times \frac{\overset{1}{\cancel{6}}}{\underset{4}{\cancel{24}}} = \frac{9}{8}$

9. Explain why each of the following attempts at "cancellation" is incorrect.

   a. $\frac{\overset{3}{\cancel{6}}}{\underset{1}{\cancel{2}}} \times \frac{\overset{3}{\cancel{6}}}{5} = \frac{9}{5}$

   c. $\frac{\overset{1}{\cancel{7}}}{\underset{1}{\cancel{4}}} + \frac{\overset{2}{\cancel{8}}}{\underset{2}{\cancel{7}}} = \frac{2}{2} = 1$

   b. $\frac{3 + \overset{1}{\cancel{2}}}{4 + \underset{1}{\cancel{2}}} = \frac{3}{4}$

10. Write the set of numbers R consisting of the reciprocals of the members of the set, Q, where

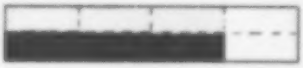
$$Q = \{1, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{6}{7}, \frac{7}{8}\}.$$

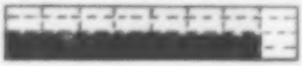
11. a. When is the reciprocal of a number greater than the number?
    b. When is the reciprocal of a number less than the number?
    c. When is the reciprocal of a number equal to the number?
    d. If n is a counting number, can we correctly say that one of the following is always true?

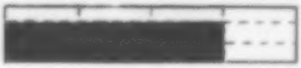
    (1) $n > \frac{1}{n}$ (2) $\frac{1}{n} > n$ (3) $n = \frac{1}{n}$

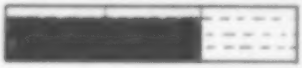
Solutions for Problems

1. The end results should look something like these:

a.  $\frac{1}{2}$ of $\frac{3}{4} = \frac{3}{8}$

b.  $\frac{2}{4}$ of $\frac{7}{8} = \frac{14}{32}$

c.  $\frac{2}{3}$ of $\frac{3}{4} = \frac{6}{12}$

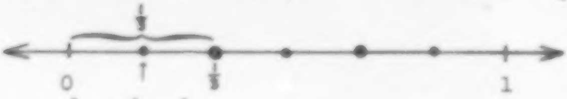
d.  $\frac{3}{4}$ of $\frac{2}{3} = \frac{6}{12}$

2. Yes. The "of" operation is commutative.

3. a. 24 b. 14

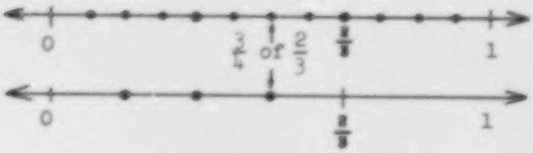
4. 9, 8; 7×9; 5×8; $\frac{5 \times 8}{7 \times 9}$

5. a. $\overline{DE}$ b. $\overline{DF}$ c. $\overline{DF}$ d. $\overline{DG}$ (since $\frac{6}{4} = \frac{3}{2}$) e. $\overline{DE}$

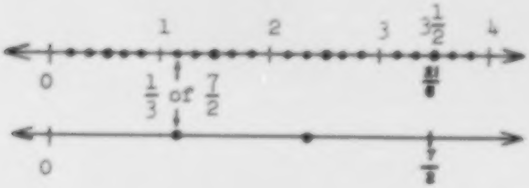
6. 

$\frac{1}{2}$ of $\frac{1}{3} = \frac{1}{6}$

7. a. $\frac{3}{4}$ of $\frac{2}{3} = \frac{6}{12}$


0
$\frac{3}{4}$ of $\frac{2}{3}$
1
0
1

b. $\frac{1}{3}$ of $3\frac{1}{2} = \frac{7}{6}$ or $1\frac{1}{6}$


1
2
3
$3\frac{1}{2}$
4
0
$\frac{1}{3}$ of $\frac{7}{2}$
0

8. a. $\frac{1}{11}$ b. $\frac{5}{1}$ c. $\frac{7}{2}$ d. $\frac{3}{50}$ e. $\frac{1}{3}$

9. a. $\frac{1}{m}$ b. $\frac{1}{s}$ c. c d. $\frac{s}{r}$ e. $\frac{3}{t}$

10. a. $\frac{2}{3}$ b. $\frac{7}{10}$

Chapter 22

DIVISION OF RATIONAL NUMBERS

Introduction

We have defined specifically for rational numbers three of the four standard binary operations on numbers. In each case we have observed that rational numbers certainly face us with different situations than whole numbers so that "new" operations must be defined. That is, "addition" for whole numbers is by no means exactly the same operation, conceptually or computationally, as "addition" for rational numbers and these differences are even more marked for "multiplication" of rational numbers.

On the other hand, some concepts do carry over and the definitions of the operations have been formulated in such a way that such standard properties as commutativity, associativity, special properties of 1 and 0, and the like, still apply, with appropriate modifications in stating them.

The pattern we will follow in discussing "division" for rational numbers will, by now, be a familiar one. Except in certain restricted instances the models and concepts discussed for division of whole numbers will not take us very far. To the extent that they do apply, however, they are suggestive of a computational way of proceeding for division of rational numbers. As before, we will want to preserve, as far as possible, the special definitions and properties that apply to division of whole numbers. Multiplication will come into the matter, as you would expect. These considerations lead us to a definition of the operation of division of rational numbers. Using this definition we can verify familiar properties of division (though most of this is left to the reader in exercises), discuss some new properties and discuss some computational procedures. Our main concern is to make clear the reasons that underlie computational rules for division of rational numbers.

A Procedure for Division of Rational Numbers

As our first link to previous work, let us consider briefly one model involving multiplication by a rational number that we did *not* use in the last chapter, principally because it is of such a limited applicability. Suppose we want to know how many objects there would be in $\frac{1}{3}$ of a set of 12 objects. This could be done by displaying the 12 objects, dividing the set into 3 equal parts, then taking one of these, as in Figure 22-1.

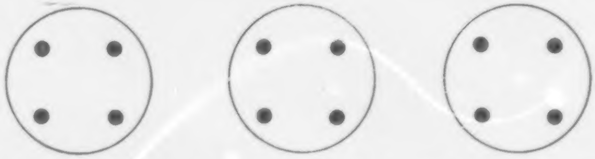


Figure 22-1. Model for $\frac{1}{3} \times 12$ and for $12 \div 3$.

Hence, $\frac{1}{3}$ of 12 is shown to be 4. Now observe that precisely the same model serves to illustrate $12 \div 3$, if this is interpreted in one of the standard ways, namely, as the number of members in each of three matching subsets of a set with 12 members. The model shows that division by 3 and multiplication by $\frac{1}{3}$ gives the same result. For reasons that will become apparent we point out that $\frac{1}{3}$ is the reciprocal of 3, as explained in Chapter 21.

As our second link to previous work let us consider the problem $6 \div \frac{1}{3}$. Now we have nothing at hand yet to even say whether or not this has any meaning, (since we don't yet have a "division" operation for rational numbers) but let us suppose that it does and state it as asking the question "How many $\frac{1}{3}$'s are there in 6?"; which will be recognized as similar to another standard way of interpreting division of whole numbers.

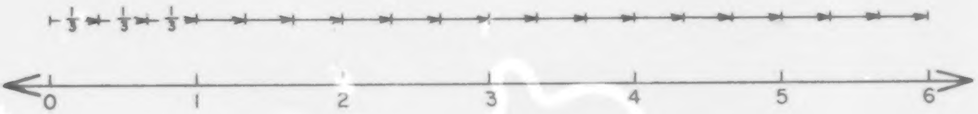


Figure 22-2. $1 \div \frac{1}{3}$ as "How many $\frac{1}{3}$'s in 6?"

Figure 22-2 shows a way of answering this question using the number line. Just by counting the segments of length $\frac{1}{3}$, we see that there are 18 of them in a segment whose length is 6. But we wouldn't need to count, for there are surely 3 such $\frac{1}{3}$ segments in each unit segment, and hence 6×3 of them in 6 units. Thus $6 \div \frac{1}{3}$ gives the same result as 6×3. We observe that 3 is the reciprocal of $\frac{1}{3}$ and note that division by $\frac{1}{3}$ is equivalent to multiplication by the reciprocal of $\frac{1}{3}$.

To move to a harder example, consider $8 \div \frac{2}{3}$, stated in the form "How many $\frac{2}{3}$'s in 8?" Again on the number line, as in Figure 22-3, we observe that it takes 12 segments of length $\frac{2}{3}$ to get a segment of length 8. We are interested in getting a computational procedure to replace this number line procedure which would surely become tedious for, say, $102 \div \frac{2}{3}$. So we observe that each unit contains $1\frac{1}{2}$ of the segments with length $\frac{2}{3}$, hence for our problem, $8 \div \frac{2}{3}$, there should be $8 \times 1\frac{1}{2} = 12$ of the $\frac{2}{3}$ segments in the 8 units as indeed there are. Expressing $1\frac{1}{2}$ as $\frac{3}{2}$, we observe that $8 \div \frac{2}{3}$ gives the same result as $8 \times \frac{3}{2}$; and that $\frac{3}{2}$ is the reciprocal of $\frac{2}{3}$.

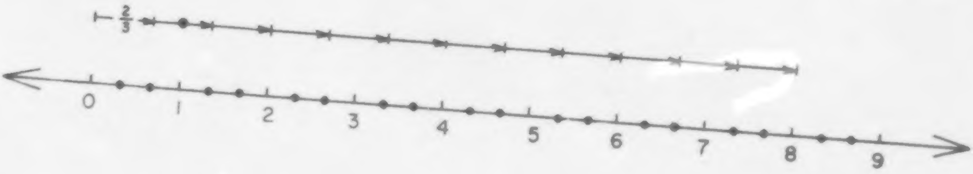


Figure 22-3. "How many $\frac{2}{3}$'s in 8?", i.e., $8 \div \frac{2}{3}$.

All this is suggestive of a way of making division of rational numbers depend on a previously defined operation: we observe that *division* by a number apparently gives the same result as *multiplication* by the reciprocal of that number. That is,

$$\frac{a}{b} \div \frac{c}{d} \text{ is equivalent to } \frac{a}{b} \times \frac{d}{c}$$

provided, of course, that none of b, c or d is zero. This will be recognized immediately in terms of the well known (and little understood) rule, "invert and multiply."

Hence the "invert and multiply" rule becomes instead a "multiply by the reciprocal of the divisor" rule. But to replace the "invert the divisor and multiply" rule by a "multiply by the reciprocal of the divisor" rule will not, of itself, result in increased understanding or avoid the misapplications that are so familiar and painful to all of us. And the "evidence" we have given so far is pretty flimsy, using only carefully selected examples, all of which have involved at least one whole number. Let us show a couple of harder examples to strengthen this *suggestion* about how to proceed, then turn to a different sort of justification. Figure 22-4a shows $\frac{8}{3} \div \frac{2}{3} = 4$, by showing that the answer to "how many $\frac{2}{3}$'s in $\frac{8}{3}$?" is 4. To test our tentative procedure we multiply $\frac{8}{3}$ times $\frac{3}{2}$, the reciprocal of $\frac{2}{3}$, and get $\frac{8}{3} \times \frac{3}{2} = \frac{24}{6} = 4$. Finally, 22-4b shows $\frac{10}{6} \div \frac{2}{3}$ as "how many $\frac{2}{3}$ in $\frac{10}{6}$?" Observe that 2 of the $\frac{2}{3}$'s arrows fall short, while 3 of them go beyond. Figure 22-4b suggests that there are $2\frac{1}{2}$ of the $\frac{2}{3}$'s segments in a $\frac{10}{6}$ segment. Doing this computation by multiplying $\frac{10}{6}$ by the reciprocal of $\frac{2}{3}$, we see

$$\frac{10}{6} \div \frac{2}{3} = \frac{10}{6} \times \frac{3}{2} = \frac{10 \times 3}{6 \times 2} = \frac{5 \times (2 \times 3)}{2 \times (3 \times 2)} = \frac{5}{2} = 2\frac{1}{2},$$

which is the same as our result on the number line.

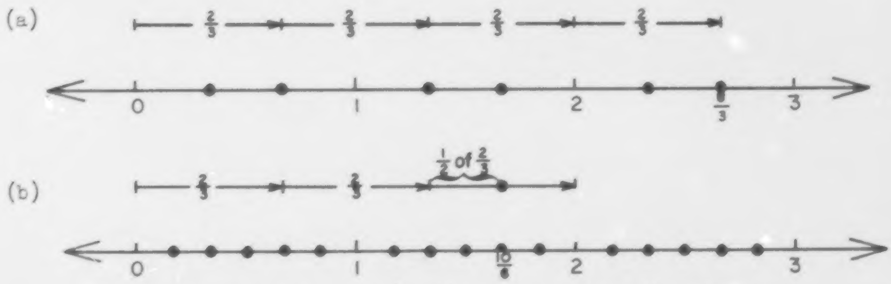


Figure 22-4. a. $\frac{8}{3} \div \frac{2}{3} = 4$; b. $\frac{10}{6} \div \frac{2}{3} = 2\frac{1}{2}$.

Problems *

1. Using models suggested by Figure 22-1, exhibit:
   a. $\frac{1}{4}$ of 12 and $12 \div 3$.
   b. $\frac{1}{2}$ of $\frac{1}{4}$ of 16 and $(16 \div 4) \div 2$.
2. If the objects in question can be cut up into smaller pieces, can such a model serve for $\frac{1}{3}$ of 7 and $7 \div 3$? If so, show it.
3. Using number line models, show solutions to the following:
   a. How many $\frac{1}{2}$'s in 7? (i.e., $7 \div \frac{1}{2}$)
   b. $2\frac{3}{4} \div \frac{1}{2}$

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* Solutions for problems in this chapter are found on page 289.

c. A pattern for curtains calls for $2\frac{1}{2}$ yards of material of a curtain width for each panel. How many panels can be made from 15 yards of material? Is $15 \div 2\frac{1}{2}$ the same as 15 times the reciprocal of $2\frac{1}{2}$?

4. a. Write a number sentence using division to solve the following: If a quart is $\frac{1}{4}$ of a gallon, how many quarts are in $5\frac{1}{2}$ gallons?

   b. What multiplication sentence could be used to solve the same problem?

   c. Do the two sentences fit the pattern that division by a number is equivalent to multiplication by its reciprocal?

The Definition for Division of Rational Numbers

We have a possible procedure for division indicated by our previous experience and our models for division of whole numbers and multiplication of rational numbers. Another tack we can take is just to lay aside, for the time being, all such "practical" considerations and formulate a definition for the operation just on the basis that it must be analogous to the formal definition of division of whole numbers. This definition was given in Chapter 9 as

$$a \div b = n \text{ if and only if } a = b \times n; \text{ where } b \neq 0.$$

We recognize this definition as essentially an instruction to "check" any supposed answer by multiplying quotient times divisor to get back (hopefully) the dividend. (Of course the "if and only if" says that one could "check" a multiplication by an appropriate division, but this is seldom done.) The corresponding statement which we can take as a definition of division of rational numbers is

$$\frac{a}{b} \div \frac{c}{d} = \frac{m}{n} \text{ if and only if } \frac{m}{n} \times \frac{c}{d} = \frac{a}{b}\text{; where none of } b, c, d \text{ or } n \text{ can be zero.}$$

You may remember that this was described in Chapter 9 as the "missing factor" concept of division. Using this definition one gets an answer for the problem $\frac{8}{3} \div \frac{2}{3} = \frac{m}{n}$ by looking for $\frac{m}{n}$ in the problem $\frac{m}{n} \times \frac{2}{3} = \frac{8}{3}$. This becomes, by our definition of multiplication of rational numbers, $\frac{m \times 2}{n \times 3} = \frac{8}{3}$ by which it is apparent that $\underline{m}$ can be 4 and $\underline{n}$ can be 1.

So $\frac{8}{3} \div \frac{2}{3} = \frac{4}{1} = 4$, as we found earlier in this chapter using the number line. Let us take another example, $\frac{4}{9} \div \frac{2}{3} = \frac{m}{n}$, which we put in the form $\frac{m}{n} \times \frac{2}{3} = \frac{4}{9}$. This can be written $\frac{m \times 2}{n \times 3} = \frac{4}{9}$, by which it is apparent that $\underline{m}$ can be 2 and $\underline{n}$ can be 3, so $\frac{4}{9} \div \frac{2}{3} = \frac{2}{3}$. But we are faced with a real dilemma if we attempt something like $\frac{2}{3} \div \frac{4}{9} = \frac{m}{n}$ for this becomes $\frac{m}{n} \times \frac{4}{9} = \frac{2}{3}$ or $\frac{m \times 4}{n \times 9} = \frac{2}{3}$. Now it is pretty hard to see what $\underline{m}$ and $\underline{n}$ should be to get the correct result, since there is no whole number times 4 that gives 2 as a product and no counting number times 9 that gives 3. If $\frac{2}{3}$ is replaced by its equivalent, $\frac{12}{18}$, the problem becomes $\frac{m}{n} \times \frac{4}{9} = \frac{12}{18}$ and the answer is now apparent, namely, $\frac{m}{n} = \frac{3}{2}$. But how would we know what equivalent fraction to use? It certainly is not obvious what one should do to find $\frac{m}{n}$ in, for example, the recasting of $\frac{3}{7} \div \frac{2}{5} = \frac{m}{n}$ in the form $\frac{m}{n} \times \frac{2}{5} = \frac{3}{7}$. There is surely some equivalent of $\frac{3}{7}$ of the form $\frac{m \times 2}{n \times 5}$ but it isn't clear immediately what it would be. (As a matter of fact, $\frac{3}{7} = \frac{30}{70}$ will do the trick, in which case $\frac{m}{n} = \frac{15}{14}$, as the reader should verify.)

It would seem that our formal definition, $\frac{a}{b} \div \frac{c}{d} = \frac{m}{n}$ if and only if $\frac{m}{n} \times \frac{c}{d} = \frac{a}{b}$, does give us a direct tie-in to the definition of division of whole numbers but does not provide us with a very good way of computing answers. Our earlier procedure $(\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c})$, on the other hand, rests on some flimsy justification but gives an immediate way to get an answer. The problem just considered, for example, becomes $\frac{3}{7} \div \frac{2}{5} = \frac{3}{7} \times \frac{5}{2} = \frac{15}{14}$, the same result as above.

For the moment let us investigate division from both angles, accepting the formal definition as *the* definition, but using the computational procedure "to divide by a number, multiply by its reciprocal," to get answers. In the course of this investigation we will show that the computational procedure *always* gives quotients that "fit" the formal definition.

An Investigation of Division

In this section we propose to indicate various ways in which a division problem involving whole numbers, rational numbers, or both might appear and indicate the patterns that result from solving such problems. In each case we are interested in a number $\frac{m}{n}$ such that $\frac{a}{b} \div \frac{c}{d} = \frac{m}{n}$.

In all of Figure 22-5 we assume that no denominator can be zero.

Computation using $\frac{a}{b} \div \frac{c}{d} = q = \frac{a}{b} \times \frac{d}{c}$	Verification using the definition $\frac{a}{b} \div \frac{c}{d} = \frac{m}{n}$ if and only if $\frac{m}{n} \times \frac{c}{d} = \frac{a}{b}$
Case 1 Two whole numbers:	
$7 \div 3 = \frac{7}{1} \div \frac{3}{1} = \frac{7}{1} \times \frac{1}{3} = \frac{7}{3}$	$\frac{7}{3} \times \frac{3}{1} = \frac{7 \times 3}{1 \times 3} = \frac{7}{1} = 7$
In general:	
$a \div c = \frac{a}{1} \div \frac{1}{c} = \frac{a}{c}$, $c \neq 0$	$\frac{a}{c} \times \frac{c}{1} = \frac{a \times c}{1 \times c} = \frac{a}{1} = a$
(Hence $\frac{a}{c}$ appears as an equivalent way to express $a \div c$)	
Case 2 Whole number ÷ rational number or rational number ÷ whole number:	
a. $7 \div \frac{3}{4} = \frac{7}{1} \times \frac{4}{3} = \frac{7 \times 4}{3} = \frac{28}{3}$	$\frac{28}{3} \times \frac{3}{4} = \frac{7 \times 4 \times 3}{3 \times 4} = \frac{7}{1} = 7$
In general:	
$a \div \frac{c}{d} = \frac{a}{1} \times \frac{d}{c} = \frac{a \times d}{c}$	$\frac{a \times d}{c} \times \frac{c}{d} = \frac{a \times (d \times c)}{(c \times d)} = \frac{a}{1} = a$
b. $\frac{3}{4} \div 7 = \frac{3}{4} \div \frac{7}{1} = \frac{3}{4} \times \frac{1}{7} = \frac{3 \times 1}{4 \times 7} = \frac{3}{28}$	$\frac{3}{28} \times \frac{7}{1} = \frac{3 \times 7}{28 \times 1} = \frac{3 \times 7}{4 \times 7} = \frac{3}{4}$
In general:	
$\frac{a}{b} \div c = \frac{a}{b} \div \frac{c}{1} = \frac{a}{b} \times \frac{1}{c} = \frac{a}{b \times c}$	$\frac{a}{b \times c} \times \frac{c}{1} = \frac{a \times c}{b \times c} = \frac{a}{b}$
Case 3 Rational number ÷ rational number:	
a. Same denominator	
$\frac{7}{8} \div \frac{3}{8} = \frac{7}{8} \times \frac{8}{3} = \frac{7 \times 8}{8 \times 3} = \frac{7}{3}$	$\frac{7}{3} \times \frac{3}{8} = \frac{7 \times 3}{3 \times 8} = \frac{7}{8}$
In general:	
$\frac{a}{b} \div \frac{c}{b} = \frac{a}{b} \times \frac{b}{c} = \frac{a \times b}{b \times c} = \frac{a}{c}$	$\frac{a}{c} \times \frac{c}{b} = \frac{a \times c}{c \times b} = \frac{a}{b}$
(The "common denominator case")	
b. Different denominators:	
$\frac{3}{4} \div \frac{5}{7} = \frac{3}{4} \times \frac{7}{5} = \frac{3 \times 7}{4 \times 5} = \frac{21}{20}$	$\frac{21}{20} \times \frac{5}{7} = \frac{21 \times 5}{20 \times 7} = \frac{(3 \times 7) \times 5}{(4 \times 5) \times 7} = \frac{(7 \times 5) \times 3}{(7 \times 5) \times 4} = \frac{3}{4}$
In general:	
$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{a \times d}{b \times c}$	$\frac{a \times d}{b \times c} \times \frac{c}{d} = \frac{(a \times d) \times c}{(b \times c) \times d} = \frac{a \times (c \times d)}{b \times (c \times d)} = \frac{a}{b}$

Figure 22-5. Various possibilities for division of numbers.

The consideration of the various possibilities exhibited in Figure 22-5 leads us very strongly to believe that the computational procedure given by " $\frac{a}{b} \div \frac{c}{d} = \frac{a}{c} \times \frac{d}{c}$ " is equivalent to the formal definition. In fact, the general statement of Case 3b suffices as the promised general "proof" that the computational procedure will *always* give a result that can be shown correct by the definition.

Case 1 suggests a new and interesting use of the fraction notation. Observe that the *division problem* $a \div c$ becomes the *fraction* $\frac{a}{c}$. If we permit ourselves to extend this practice to include *any* division problem, we get fractions that no longer resemble those we are used to. For example, $\frac{2}{3} \div \frac{3}{4}$ would become $\dfrac{\frac{2}{3}}{\frac{3}{4}}$; $14\frac{1}{2} \div 3\frac{1}{5}$ would become $\dfrac{14\frac{1}{2}}{3\frac{1}{5}}$; and so on. Such a more generalized use of the fraction notation is a very common one, especially in algebra, so that we should not become too attached to the idea that fractions are *always* of the simple form $\frac{a}{b}$ that we have used in this book up to now. Furthermore, while it is true that rational numbers can always be expressed in a fractional form, with a whole number above the line and a counting number below the line; it is *not* true that every fraction with *something* written above the line and something written below the line represents a rational number. Still another common use for the fractional form is with ratios and proportions, as will be discussed in Chapter 24. In cases where they are used, the fractional forms behave much *as if* they represented rational numbers, but their conceptual interpretation may be quite different. For example, division may be represented and manipulated as a fraction form, but the conception of division as an operation is quite different from the conception of rational numbers as numbers.

The common denominator situation, Case 3a, is also of interest since the quotient of $\frac{a}{b} \div \frac{c}{b}$ is $\frac{a}{c}$ and if this fraction form is interpreted as a division, as just discussed, the division of two such rational numbers ends up as a division of the two whole number numerators. This suggests another method of making division of rational numbers depend on already known operations, since one can always change the dividend and divisor to fractions with a common denominator. Also, if we state, say, $\frac{6}{10} \div \frac{2}{10}$ as "how many $\frac{2}{10}$ in $\frac{6}{10}$?" it somehow "seems right" (at least to many people to find the answer as $6 \div 2$. Use of the pattern shown by the common denominator case might also help in explaining why $.2\overline{).6}$ gives the same answer as $2\overline{)6}$ (the result of "moving the decimal point") and $.04\overline{).32}$ ($\frac{32}{100} \div \frac{4}{100}$) gives the same result as $4\overline{)32}$.

Observe that our examination of cases reveals that any division involving whole numbers and/or rational numbers can now be performed. This is the really "new" contribution of the division of rational numbers operation. While for whole numbers we have had to caution that division was not "closed" since, for example, $3 \div 7$ does not have a whole number answer, we have now enlarged our set of numbers and defined the division operation so that all such problems do have answers; in this case $\frac{3}{7}$. Furthermore, such problems as $7 \div 3$ need no longer be stated as having a "quotient" 2 and "remainder" 1, for $7 \div 3$ is now $\frac{7}{3}$ or $2\frac{1}{3}$. It is at this point that it becomes "legal," in the mathematical scheme of things, to use the remainder from a division problem along with the divisor to form a fraction to be included as part of the quotient. Hence, $37 \div 15$ can be regarded as having an exact "quotient," $2\frac{7}{15}$, rather than a "quotient" 2 and "remainder" 7. The corresponding "checks" consist of the multiplication $2\frac{7}{15} \times 15 = 37$ in the first instance rather than the division algorithm $37 = (2 \times 15) + 7$ in the second instance. Observe that "quotient" is used in two senses here, with the second use being that described earlier as a "missing factor" in the requirement of the definition that $a \div b$ is the number n such that $n \times b = a$. Our situation now is that such a missing factor exists for every division problem, with the usual exclusion of zero divisors.

Problems

5. For each of the following tell which "case" in Figure 22-5 identifies the pattern.

   a. $7 \div \frac{3}{5} = \frac{7 \times 5}{3}$

   b. $7 \div 5 = \frac{7}{5}$

   c. $\frac{7}{9} \div \frac{5}{9} = \frac{7}{5}$

   d. $\frac{3}{5} \div 7 = \frac{3}{7 \times 5}$

6. For each of the following divisions, $\frac{a}{b} \div \frac{c}{d}$, tell whether the result is $a \div c$.

   a. $\frac{7}{10} \div \frac{3}{10}$

   b. $\frac{7}{100} \div \frac{3}{10}$

   c. $\frac{1}{100} \div \frac{7}{1000}$

   d. $\frac{5}{8} \div \frac{3}{8}$

   e. $\frac{3}{8} \div \frac{8}{5}$

7. For each of the above that does not result directly in $a \div c$, first get an answer in lowest terms using the pattern $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$ then change the fractions to fractions with common denominators so that the pattern $\frac{a}{b} \div \frac{c}{b} = \frac{a}{c}$ can be used and show the answer obtained in this way.

8. Find the rational number quotient in each of the following (not a quotient and remainder).

   a. $633 \div 14$ b. $1070 \div 51$ c. $47 \div 45$

As a final way of justifying the statement that dividing one rational number by another is equivalent to multiplying the dividend by the reciprocal of the divisor, let us write such a division problem in fractional form and assume that we know that such fractional forms have the properties shown in Figure 22-6 no matter what is written in the $\triangle$ above the line or the $\square$ below the line (the usual cautions about zero still apply of course).

Property A

$$\frac{\triangle}{1} = \triangle$$

Property B

$$\frac{\triangle}{\square} = \frac{h \times \triangle}{h \times \square}$$

Figure 22-6. General equivalence properties.

Figure 22-7 outlines the solution to such a division problem.

Step 1. $\frac{2}{3} \div \frac{5}{7} = \frac{\frac{2}{3}}{\frac{5}{7}}$

Step 1. Write the division in fraction form.

Step 2. $\frac{\frac{2}{3}}{\frac{5}{7}} = \frac{\frac{2}{3} \times \frac{7}{5}}{\frac{5}{7} \times \frac{7}{5}}$

Step 2. We are interested in disposing of the denominator by making it equivalent to 1 and applying property A. To do this we apply property B using the reciprocal of $\frac{5}{7}$ as the multiplier $\underline{h}$.

Step 3. $\frac{5}{7} \times \frac{7}{5} = 1$ So

$$\frac{\frac{2}{3} \times \frac{7}{5}}{\frac{5}{7} \times \frac{7}{5}} = \frac{\frac{2}{3} \times \frac{7}{5}}{1} = \frac{2}{3} \times \frac{7}{5}$$

Step 3. Here we have applied the reciprocal property and property A, with $\frac{2}{3} \times \frac{7}{5}$ occupying the Δ in that property.

Hence, $\frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \times \frac{7}{5}$

Step 4. This simply writes the problem we started with and the end result in a single sentence.

Written on one line and in a general form this becomes:

$$\frac{a}{b} \div \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{\frac{a}{b} \times \frac{d}{c}}{\frac{c}{d} \times \frac{d}{c}} = \frac{\frac{a}{b} \times \frac{d}{c}}{1} = \frac{a}{b} \times \frac{d}{c}$$

Figure 22-7. Another way of showing that $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$.

Properly presented, this method provides a convincing way of giving meaning to the "invert and multiply" rule (or, in our terms, the "multiply by the reciprocal of the divisor" rule). As we observed earlier in this chapter, the usual properties of fractions of the form $\frac{a}{b}$ ($\underline{b}$ a counting number, $\underline{a}$ a whole number) also apply in a wide variety of other uses of the fraction form, including, as in this case, fractions with fraction numerators and fraction denominators.

Problem

9. Show in some detail the solution to each of the following using the fraction form as illustrated in Figure 22-7.

a. $\frac{2}{3} \times \frac{5}{7}$

b. $\frac{31}{2} \div \frac{2}{3}$

c. $4\frac{1}{2} \div \frac{3}{7}$

d. $\frac{1}{2} \div 1\frac{1}{2}$

Summary

In this chapter we started by using analogies with the conceptual ways of describing division of whole numbers to suggest a way of getting a computing procedure for division of rational numbers. We then turned to a formal definition of such a new "division" by formulating a definition exactly similar to the definition of division of whole numbers, which essentially says that one has the correct quotient if multiplying the quotient times the divisor gives back the dividend. We showed by the consideration of a number of "cases" that the computational procedure we arrived at always gives a result that fits this definition. "Closure" of the rationals under this operation of division was considered. Different uses of the fraction form were investigated, as suggested by the result $a \div c = \frac{a}{c}$, which suggests writing division problems as dividend over the divisor rather than using the "÷" symbol. It was pointed out that this way of indicating division is the usual one in mathematics beyond arithmetic.

We did not go to the trouble here of verifying that other expected properties of division still hold. We merely state that essentially the same things are true with respect to non-commutativity, non-associativity, and the existence of a special right hand distributive pattern as one would expect from the discussion of Chapter 9. Some of these matters are included in the exercises for this chapter.

Exercises - Chapter 22

1. As with division of whole numbers, division of rational numbers is in general not commutative and not associative. Also, 1 does act as an identity element on the right, i.e., $\frac{a}{b} \div 1 = \frac{a}{b}$ (but $1 \div \frac{a}{b} = \frac{b}{a} \neq \frac{a}{b}$); there is a right hand distributive pattern (but not a left hand distributive pattern); and $0 \div \frac{a}{b} = 0$ (but $\frac{a}{b} \div 0$ is meaningless). For each of the following sentences, first see whether it is true or false. If it is true and exhibits one of the properties above that does hold, name that property. If it is false, but indicates a certain pattern that is not a property of division, indicate what property (e.g., "division is not commutative").

   a. $\frac{3}{4} \div \frac{2}{3} = \frac{2}{3} \div \frac{3}{4}$

   b. $\frac{1}{2} \div (\frac{3}{4} \div \frac{2}{3}) = (\frac{1}{2} \div \frac{3}{4}) \div \frac{2}{3}$

   c. $\frac{1}{2} \div (\frac{3}{4} \div \frac{2}{3}) = (\frac{3}{4} \div \frac{2}{3}) \div \frac{1}{2}$

   d. $\frac{1}{2} \div (\frac{3}{4} \div \frac{2}{3}) = \frac{1}{2} \div (\frac{2}{3} \div \frac{3}{4})$

   e. $\frac{3}{4} \div 1 = \frac{3}{4}$

   f. $(\frac{3}{4} + \frac{2}{3}) \div \frac{1}{2} = (\frac{3}{4} \div \frac{1}{2}) + (\frac{2}{3} \div \frac{1}{2})$

   g. $\frac{1}{2} \div (\frac{3}{4} + \frac{2}{3}) = (\frac{1}{2} \div \frac{3}{4}) + (\frac{1}{2} \div \frac{2}{3})$

   h. $(\frac{6}{7} - \frac{4}{5}) \div \frac{1}{2} = \frac{3}{7} - \frac{2}{5}$

   i. $\frac{0}{7} \div \frac{2}{7} = 0$

   j. $0 \div \frac{2}{7} = 0$

   k. $(\frac{2}{3} \times \frac{5}{7}) \div \frac{5}{7} = \frac{2}{3}$

   l. $(\frac{2}{3} \div \frac{5}{7}) \times \frac{5}{7} = \frac{2}{3}$

   m. $\frac{1}{3} \div \frac{2}{5}$ is a rational number

   n. $(\frac{2}{3} \times \frac{0}{7}) \div \frac{0}{7} = \frac{2}{3}$

   o. $\frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \times \frac{7}{5}$

2. a. $\frac{12}{10}$ is how many times as large as $\frac{12}{100}$?

   b. $\frac{4}{7}$ is how many times as large as $\frac{7}{4}$?

3. Fill in each blank below with ">", "<" or "=" to make a true mathematical sentence.

   a. $2\frac{1}{2} \times 3\frac{1}{4}$ ___ $3\frac{1}{4} \times 2\frac{1}{2}$

   b. $\frac{5}{2} \times \frac{2}{5}$ ___ $\frac{2}{3} \times \frac{2}{3}$

   c. $6 \div \frac{4}{5}$ ___ $3 \div \frac{2}{5}$

   d. $2\frac{2}{3} \times \frac{1}{4}$ ___ $4 \div \frac{3}{4}$

   e. $5 \div 3\frac{1}{2}$ ___ $1 \div \frac{2}{3}$

   f. $8 \div \frac{4}{3}$ ___ $1 \div \frac{1}{6}$

   g. $\frac{2}{3} \div \frac{7}{8}$ ___ $\frac{2}{3} \times \frac{7}{8}$

   h. $1\frac{1}{2} \div \frac{3}{3}$ ___ $3 \div \frac{3}{4}$

4. In Chapter 21 we stated the multiplicative inverse (or reciprocal) property as $a \times \frac{1}{a} = 1$ if $\underline{a}$ is a counting number and $\frac{a}{b} \times \frac{b}{a} = 1$ for rational numbers. In fact, $\frac{1}{r}$ is often called the reciprocal of $\underline{r}$ even when $\underline{r}$ is a fraction or mixed numeral. By this convention, $\frac{1}{\frac{5}{2}}$ would be the reciprocal of $\frac{5}{2}$. On the other hand, by the definition in Chapter 21, the reciprocal of $\frac{5}{2}$ would be $\frac{2}{5}$. Use properties A and B from Figure 22-6 to demonstrate that $\frac{2}{5}$ names the same number as $\frac{1}{\frac{5}{2}}$.

5. Using properties A and B of Figure 22-6, show that in general, $\frac{b}{a} = \frac{1}{\frac{a}{b}}$, hence either can be regarded as the reciprocal of $\frac{a}{b}$.

6. Show using equivalent fractions that the reciprocal of $2\frac{1}{2}$ could be written as $\frac{1}{2\frac{1}{2}}$.

7. Write the reciprocal of each of the following numbers in two forms; first as $\frac{1}{r}$ and then as a fraction $\frac{a}{b}$.

a. $\frac{2}{3}$

b. $1\frac{1}{2}$

c. 22

d. $\frac{11}{9}$

e. $27\frac{1}{2}$

f. $\frac{1}{\frac{1}{2}}$

8. For each of the following, tell whether the statement is true or false.

a. $\frac{\frac{2}{3}}{\frac{5}{7}} = \frac{\frac{2}{3} \times \frac{7}{5}}{\frac{5}{7} \times \frac{7}{5}}$

b. $\frac{\frac{2}{3}}{\frac{5}{7}} = \frac{\frac{2}{3} \times \frac{5}{7}}{\frac{5}{7} \times \frac{5}{7}}$

c. $\frac{\frac{2}{3}}{\frac{5}{7}} = \frac{\frac{2}{3} \times \frac{5}{7}}{\frac{5}{7} \times \frac{7}{5}}$

d. $\frac{\frac{2}{3}}{\frac{5}{7}} = \frac{2}{3} \times \frac{5}{7}$

e. $\frac{\frac{2}{3}}{\frac{5}{7}} = \frac{2}{3} \times \frac{7}{5}$

f. $\frac{\frac{2}{3}}{\frac{5}{7}} = \frac{7}{5} \times \frac{2}{3}$

g. $\frac{\frac{2}{3}}{\frac{5}{7}} = \frac{\frac{2}{3} \times 7 \times 3}{\frac{5}{7} \times 7 \times 3}$

9. Show how the pattern shown in Exercise (g) above can show that

a. $\frac{2}{3} \div \frac{5}{7} = \frac{14}{15}$ b. $\frac{3}{5} \div \frac{2}{3} = \frac{9}{10}$

10. A jet plane made a flight from San Francisco to New York in $4\frac{1}{4}$ hours. This was $\frac{5}{7}$ of the time it took a turbo-prop plane. How many hours did the turbo-prop require for the trip?

11. Find $18,375 \div 25$. Tell why this can be done quickly by multiplying 18,375 by 4 and dividing by 100. (Recall that $25 = \frac{100}{4}$.)

12. Mr. Jones received 16 shares of stock on a stock split which gave $\frac{2}{3}$ of a share in dividend for every share held. How many shares did Mr. Jones have before the split?

13. For each of the following determine whether it is possible to express the number as an equivalent fraction such that the numerator is a whole number and the denominator is 10 or 100 or 1000.

a. $\frac{4}{5}$

b. $\frac{3}{8}$

c. $\frac{2}{3}$

d. $\frac{5}{6}$

e. $\frac{4}{9}$

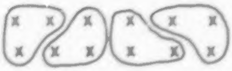
f. $\frac{0}{3}$

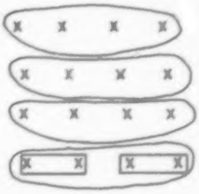
g. $\frac{3}{50}$

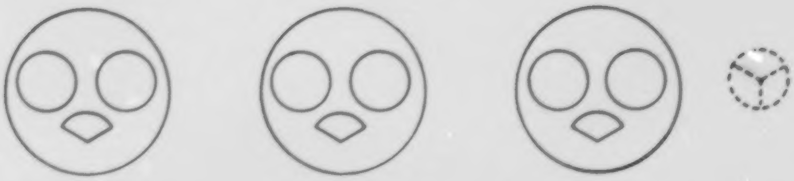
h. $\frac{7}{125}$

14. Study the various factors of each of the above denominators; when can you be sure it would be possible to express a rational number as an equivalent fraction such that the numerator is a whole number and the denominator is a power of 10?

Solutions for Problems

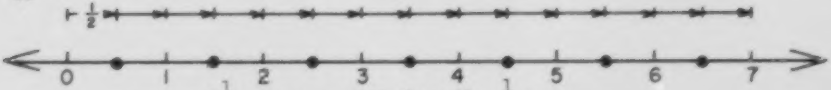
1. a.  $\frac{1}{4}$ of $12 = 12 \div 4 = 3$

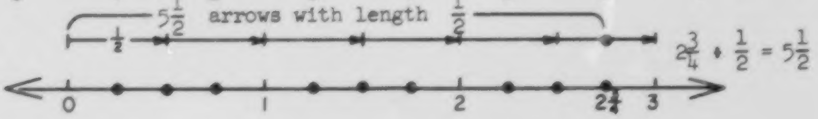
b.  $\frac{1}{2}$ of $\frac{1}{4}$ of $16 = (16 \div 4) \div 2 = 2$

2. 

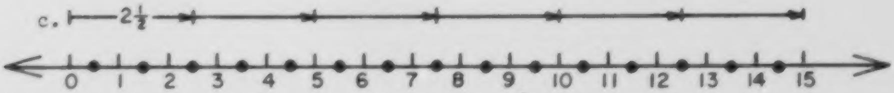
$\frac{1}{3}$ of $7 = 7 \div 3 = 2\frac{1}{3}$

3. a. There are fourteen $\frac{1}{2}$'s in 7.


$\frac{1}{2}$
0
1
2
3
4
5
6
7

b. 
$5\frac{1}{2}$ arrows with length $\frac{1}{2}$
$\frac{1}{2}$
0
1
2
$2\frac{3}{4}$
3

$2\frac{3}{4} \div \frac{1}{2} = 5\frac{1}{2}$

c. 
$2\frac{1}{2}$
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15

6 panels. Yes, since $15 \div \frac{5}{2} = \frac{15}{1} \times \frac{2}{5} = 6$.

4. a. $5\frac{1}{2} \div \frac{1}{4} = 22$

b. $5\frac{1}{2} \times 4 = \frac{11}{2} \times \frac{4}{1} = 22$ (If a quart is $\frac{1}{4}$ gallon, there are 4 of them in each gallon.)

c. Yes

5. a. Case 2a c. Case 3a
b. Case 1 d. Case 2b

6. a. Yes b. No c. No d. Yes e. No

7.

b. $\frac{7}{\underset{10}{\cancel{100}}} \times \frac{\overset{1}{\cancel{10}}}{3} = \frac{7}{30}$; $\frac{7}{100} \div \frac{30}{100} = \frac{7}{30}$

c. $\frac{1}{100} \times \frac{1000}{7} = \frac{1000}{700} = \frac{10}{7}$; $\frac{10}{1000} \div \frac{7}{1000} = \frac{10}{7}$

e. $\frac{3}{8} \times \frac{5}{8} = \frac{15}{64}$; $\frac{15}{40} \div \frac{64}{40} = \frac{15}{64}$

8. a. $45\frac{3}{14}$ b. $20\frac{50}{51}$ c. $\frac{47}{45} = 1\frac{2}{45}$

9.

a. $$\frac{\frac{2}{3}}{\frac{5}{7}} = \frac{\frac{2}{3} \times \frac{7}{5}}{\frac{5}{7} \times \frac{7}{5}} = \frac{\frac{2}{3} \times \frac{7}{5}}{1} = \frac{2}{3} \times \frac{7}{5} = \frac{14}{15}$$

b. $$\frac{\frac{31}{2}}{\frac{2}{3}} = \frac{\frac{31}{2} \times \frac{3}{2}}{\frac{2}{3} \times \frac{3}{2}} = \frac{\frac{31}{2} \times \frac{3}{2}}{1} = \frac{31}{2} \times \frac{3}{2} = \frac{93}{4}$$

c. $$\frac{4\frac{1}{2}}{\frac{3}{7}} = \frac{4\frac{1}{2} \times \frac{7}{3}}{\frac{3}{7} \times \frac{7}{3}} = \frac{4\frac{1}{2} \times \frac{7}{3}}{1} = \frac{9}{2} \times \frac{7}{3} = \frac{63}{6}$$

d. $$\frac{\frac{1}{2}}{1\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{3}{2}} = \frac{\frac{1}{2} \times \frac{2}{3}}{\frac{3}{2} \times \frac{2}{3}} = \frac{\frac{1}{2} \times \frac{2}{3}}{1} = \frac{1}{\underset{1}{\cancel{2}}} \times \frac{\overset{1}{\cancel{2}}}{3} = \frac{1}{3}$$

Chapter 23

DECIMALS

Introduction

In the last few chapters we have considered the rational numbers named as fractions in the form $\frac{a}{b}$, with $\underline{a}$ a whole number and $\underline{b}$ a counting number, and we have discussed ways of computing with such numbers, chiefly by manipulation of their fractional forms. Another common way of naming rational numbers, as you know, is by decimals, sometimes called decimal fractions. This chapter considers this way of naming rational numbers, the operations using these numerals and the justification of "rules" that are commonly stated for doing such operations. A couple of new issues raised by this way of writing numbers are also discussed.

Decimal fractions force themselves on the attention of youngsters very early because of their use in our monetary system. More important is the fact that decimal notation is used in virtually all technical, scientific and business computing. And, as will be discussed in Chapter 30, decimals provide the only convenient means we have of dealing with certain numbers that cannot be named with a fraction in the form $\frac{a}{b}$. For the moment, however, we will regard our decimals as naming numbers which could just as well be named by whole numbers, fractions, or mixed numerals. Most of our discussion will deal with "terminating decimals" and their fraction equivalents, for example, $.7 = \frac{7}{10}$, $.78 = \frac{78}{100}$, and so on. Near the end of the chapter we will discuss some "repeating decimals" and their fractions equivalents, for example, $\frac{1}{3} = .3333 \ldots$.

Meaning of the Decimal Notation

As noted above, the point of view of this chapter principally regards decimals as another way, a "shorthand" way, if you like, of naming certain rational numbers; in particular those named by ordinary terminating decimals whose fraction forms would have denominators of 10 or 100 or 1000 or some other suitable multiple of 10. (Such products as 10×10, $10 \times 10 \times 10$, etc. that involve only tens we will call powers of ten.) The explanations given for the well known computational procedures involving such decimals will rely on corresponding procedures with fractions. That is, .1 is just another name for $\frac{1}{10}$, .27 for $\frac{27}{100}$, 4.314 for $4\frac{314}{1000}$, and so on. For such decimals all "denominators" are powers of 10, and since the particular denominator in question is revealed by the way the number is written, we just omit writing it. What needs to be settled is why the way a number is written reveals what "denominator" is involved.

We begin by recalling the expanded notation for a whole number using the base ten and the idea of place value. Thus:

$$3842 = (3 \times 1000) + (8 \times 100) + (4 \times 10) + (2 \times 1).$$

In our base ten place value system each digit represents a certain value according to its place in the numeral. In the above example, the 3 is in the thousands place, the 8 is in the hundreds place, and so on.

The whole idea of the place value notation (with base ten) is that the value of each place immediately to the *left* of a given place is *ten times* the value of the given place. But then the value of a place immediately to the *right* must be *one-tenth* of the value of the given place. To make our place value system serve for naming rational numbers as well as whole numbers we simply extend this idea of place value by saying that there are places to the *right* of the one's place and that the value attached to each place will, as before, be one-tenth that of the value of the place immediately to its left. When writing whole numbers the *last* place of the whole number was the one's place, but in writing decimals for fractions we have to fix where the one's place is with a dot (.), placed after the one's place. We will call this dot a "*decimal point*" reserving the word "decimal" for the actual numeral. Hence, the place value of the first place to the right of the one's place is $\frac{1}{10}$ of $1 = \frac{1}{10}$, that of the second, $\frac{1}{10}$ of $\frac{1}{10} = \frac{1}{100}$, that of the third, $\frac{1}{10}$ of $\frac{1}{100} = \frac{1}{1000}$, and so on. Figure 23-1 illustrates this and shows the names of each of five positions to the right and to the left of the decimal point. Each of the positions to the right we will call a *decimal place*. A terms such as "four place decimal" is meant to designate a numeral with four places to the right of the decimal point.

Hundred thousand	Ten thousand	Thousand	Hundred	Ten	One	Tenth	Hundredth	Thousandth	Ten-thousandth	Hundred-thousandth
$10\times10\times10\times10\times10$ or $10\times10{,}000$	$10\times10\times10\times10$ or $10\times1{,}000$	$10\times10\times10$ or 10×100	10×10	10×1	1	$\frac{1}{10}$	$\frac{1}{10\times10}$	$\frac{1}{10\times10\times10}$ or $\frac{1}{10}\times\frac{1}{100}$	$\frac{1}{10\times10\times10\times10}$ or $\frac{1}{10}\times\frac{1}{1000}$	$\frac{1}{10\times10\times10\times10\times10}$ or $\frac{1}{10}\times\frac{1}{10{,}000}$
100,000	10,000	1,000	100	10	1	0.1	0.01	0.001	0.0001	0.00001

Figure 23-1. Place value chart.

Hence, the numeral 435.268 expanded according to place value would be:

$$(4\times100) + (3\times10) + (5\times1) + (2\times\frac{1}{10}) + (6\times\frac{1}{100}) + (8\times\frac{1}{1000}) .$$

Such a numeral would be read as "four hundred thirty-five and two hundred sixty-eight thousandths." Observe that the "and" serves to designate the decimal point. Careless use of "and" in reading numerals is quite common and sometimes leads to confusion, as indicated by Problem 3 below. Observe also that just as we say "four hundred thirty-five" instead of "four hundreds, three tens, and five ones" we say "two hundred sixty-eight thousandths" rather than "two-tenths, six-hundredths, and eight-thousandths." The place value of the final digit tells whether we should say "tenths," "hundredths," or what have you. Finally, observe the symmetry about the ones place (not around the decimal point):

> tens one place to the left of one and tenths one place to the right of one; hundreds two places to the left of one and hundredths two places to the right of one; and so on.

Problems *

1. Write the decimal numeral for
   a. $(9\times1000) + (8\times100) + (7\times10) + (6\times1) + (5\times\frac{1}{10}) + (4\times\frac{1}{100}) + (3\times\frac{1}{1000})$
   b. $(7\times100) + (2\times1) + (9\times\frac{1}{10}) + (7\times\frac{1}{1000})$

---

* Solutions for the problems in this chapter are on page 313.

2. Write in expanded form:

   a. 927.872 b. 40.09 c. 4.00006

3. Regarding the "and" in each of the following as marking the decimal point, write the following as decimal numerals. Then write one or more numerals that one might get from a careless use of "and."

   a. Four hundred and sixty-one thousandths

   b. Two thousand three hundred and forty ten-thousandths

Equivalence and Order for Decimals

You will recall that in our models of rational numbers the denominator of a fraction denotes how many congruent parts a unit segment or region is divided into while the numerator tells how many of these parts are to be considered. If we regard the "denominator" for a decimal as being implicit in the situation though not explicitly written, so that, for example, .5 means 5 of 10 parts and .38 means 38 of 100 parts, it is easy to justify one very handy property of decimals. This property is exemplified by the fact that $.3 = .30 = .300 = .3000$ and $.37 = .370 = .3700 = .370000$ and is simply that you can put as many zeros on the end of a decimal as you like and still have equivalent decimals. Writing these examples in terms of their fraction equivalents, we get $\frac{3}{10} = \frac{30}{100} = \frac{300}{1000} = \frac{3,000}{10,000}$ and $\frac{37}{100} = \frac{370}{1000} = \frac{3,700}{10,000} = \frac{370,000}{1,000,000}$ which is clearly just a matter of multiplying both numerator and denominator by the same number. On the other hand, we can take off as many zeros from the end of a decimal as we wish, as can be seen by rewriting the above as $.3000 = .300 = .30 = .3$. This can be likened to "reducing" fractions. For decimal numbers, then, equivalence is immediately evident. Furthermore, changing two decimals to a "common denominator" is no problem at all; one simply tacks on the required number of zeros. For example, we get a common denominator for .47 (that is, $\frac{47}{100}$) and .5387 $(\frac{5,387}{10,000})$ by simply tacking two zeros onto .47 to make it .4700 $(\frac{4,700}{10,000})$.

To tell when one of a pair of numbers is "greater than" or "less than" the other number is also no problem. One simply gives them the same "denominator," in the sense explained above, and compares them directly. For example, .0387 is clearly less than .2 since we can compare .0387 $(\frac{387}{10,000})$ with .2000 $(\frac{2,000}{10,000})$.

Problems

4. Put in the proper symbol <, =, > to make a true statement:

a. 0.47 ___ .0838 c. 1.7 ___ 1.70000

b. $2\frac{1}{4}$ ___ 2.2 d. 4.5 ___ 4.49

5. Arrange from least to greatest:

.25, 2.25, 1, 0.02, 1.02, 2.002, 2.2

Operations Using Decimals

Each time we have introduced a new set of numbers, or in this case a new way of writing familiar numbers, we have developed ways of dealing with equivalence, less than or greater than relations, and we have defined ways of doing standard operations. Equivalence and order for decimals have just been dealt with. To begin a discussion of operations, let us remind ourselves that any such discussions should provide both conceptual models for the process at hand and efficient computational procedures. The conceptual aspect of the operations using terminating decimals can be very quickly disposed of by remarking that since they are only different ways of writing rational numbers, exactly the same models that were used for fractions suffice to give meaning to the operations with decimals. That is to say that for each such decimal used in an operation there is an exactly equivalent fraction of the form $\frac{a}{b}$, where b is some power of 10, so that using these equivalent fractions, the models and concepts previously discussed will apply. For that matter, any operation could be done merely by changing the decimals to fractions and using the computational procedures already discussed. It is convenient, however, to have ways of dealing directly with decimals via the usual operations.

We cannot dispose of these computational procedures very easily because though we have fairly simple rules of thumb to tell us how to get answers, these rules are seldom well understood and are often incorrectly applied. This is especially true for the operation of division. In the remainder of this section we will consider each operation in turn by stating the commonly accepted procedure for getting an answer, then explaining why it is that this procedure works, as justified by the various properties and procedures that we have listed for operations with whole numbers and rational numbers and for equivalent decimals. In most cases the procedure will amount to first a computation with whole numbers, then some rule to place the decimal point in the answer that results.

Addition of Two or More Decimals

Procedure: (a) Add enough zeros to each decimal so that all of them have the same number of decimal places.

(b) Forgetting about the decimal point, add them as if they were whole numbers.

(c) Place the decimal point so that the resulting sum has the same number of decimal places as each of the numerals in (a).

Example: (1) $34.8 + .008 + 73.74 + 147 =$

$34.800 + .008 + 73.740 + 147.000 = 255.548$

(2)

$$\begin{array}{r} 34.800 \\ .008 \\ 73.740 \\ \underline{147.000} \\ 255.548 \end{array}$$

Justification: Instruction (a) is simply an instruction to write the decimal parts of each numeral with a "common denominator," in the sense explained earlier.

Instruction (b) has the effect of adding "numerators" of the decimal parts of the mixed numerals, "carrying" the excess from these numerators over into the whole number part of the addition and adding the whole number parts.

Instruction (c) simply says that the "denominator" for the sum is the same as the common denominator of the addends.

Close examination of this problem computed using the fraction equivalents of the decimal numerals should make each of these points clear. This computation is exhibited in some detail below. Notice the "carry" from the fractional to the whole number part in the next to last step.

$$\begin{aligned}
34.8 + .008 + 73.74 + 147 &= 34\tfrac{8}{10} + \tfrac{8}{1000} + 73\tfrac{74}{100} + 147 \\
&= 34\tfrac{800}{1000} + \tfrac{8}{1000} + 73\tfrac{740}{1000} + 147 \\
&= 34 + 73 + 147 + \tfrac{800}{1000} + \tfrac{8}{1000} + \tfrac{740}{1000} \\
&= 34 + 73 + 147 + \tfrac{800 + 8 + 740}{1000} \\
&= 34 + 73 + 147 + \tfrac{1548}{1000} \\
&= 34 + 73 + 147 + \tfrac{1000}{1000} + \tfrac{548}{1000} \\
&= 34 + 73 + 147 + 1 + \tfrac{548}{1000} \\
&= 255 + \tfrac{548}{1000} = 255\tfrac{548}{1000}
\end{aligned}$$

Subtraction

The procedure for subtraction of decimals is exactly analogous to that of addition. The example worked out in Figure 23-2 should make this clear:

Using Decimals

$$\begin{array}{rr}
7.58 = & 7.580 \\
5.689 = & \underline{5.689} \\
 & 1.891
\end{array}$$

Using Fractions

$$\begin{array}{rl}
7.580 = 7\tfrac{580}{1000} = 6 + \tfrac{1000}{1000} + \tfrac{580}{1000} & = 6 + \tfrac{1580}{1000} \\
5.689 = 5\tfrac{689}{1000} = 5 \quad + \quad \tfrac{689}{1000} & = \underline{5 + \tfrac{689}{1000}} \\
 & \quad 1 + \tfrac{891}{1000} = 1\tfrac{891}{1000}
\end{array}$$

Figure 23-2. 7.58 - 5.689.

Observe that the regrouping necessary when using "fractions" is taken care of by the ordinary subtraction of whole numbers in the problem using decimals.

Problem

6. Do each of the following problems first using the decimals, second using mixed numeral equivalents of the decimals, and third using fraction equivalents of the decimals. Do not reduce the answers.
   a. 8.9 + 3 + 5.375
   b. Subtract 12.57 from 40.

Multiplication

Procedure: (a) Pretend for the moment that the decimal points do not exist and do an ordinary multiplication as if only whole numbers were involved.

(b) Place the decimal point in the resulting product by merely counting the number of decimal places in each factor, adding these two counts, and putting the decimal point so that the product has this many decimal places. If there are not enough digits in the product to accomodate the required number of decimal places, one must supply zeros between the decimal point and the first digit of the whole number product. Figure 23-3 illustrates this process.

Examples:

Using Decimals

(a) .5 × .73

5 × 73 = 365

Three decimal places are required, so

.5 × .73 = .365

Using Fractions

$$\frac{5}{10} \times \frac{73}{100} = \frac{5 \times 73}{10 \times 100} = \frac{365}{1000}$$

(b) 2.1 × .032

$$\begin{array}{r} 32 \\ \times\ 21 \\ \hline 32 \\ 64 \\ \hline 672 \end{array}$$

Four decimal places are required, so

2.1 × .032 = .0672

$$2\frac{1}{10} \times \frac{32}{1000} = \frac{21}{10} \times \frac{32}{1000} = \frac{21 \times 32}{10 \times 1000}$$

$$= \frac{672}{10,000}$$

Figure 23-3. Multiplication of decimals.

Justification: Examination of the problems using fractions in Figure 23-3 shows that in each such problem we always end up getting a numerator for the product by multiplying exactly the whole numbers that one gets by pretending that the decimal points don't exist. This justifies the first part of our procedure. Likewise, in the fraction problem, we end up multiplying, for example, tenths times hundredths to get thousandths (so a one place decimal times a two place decimal requires a three place decimal as the product); tenths times thousandths to get ten-thousandths (so a one place decimal times a three place decimal requires a four place decimal as the product); and so on. To consider more possibilities, hundredths (two decimal places) times hundredths (two decimal places) would give ten-thousandths (four decimal places), hundredths times thousandths (three decimal places) gives hundred-thousandths (five decimal places) and so on.

Perhaps the best way to convince oneself of the validity of the procedures used in multiplying decimals, or alternatively to "discover" what rules might work, is to work a number of such problems using the fraction equivalents of the decimals and the definition $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$. In this case observe that one does not work with the numbers in "simplest form" but must retain the denominators as powers of ten.

Problems

7. Make each of the following a true statement by supplying the missing decimal point and missing zeros in factor or product as required.

   a. .7 × .9 = 63
   b. 7 × .02 = .014
   c. .006 × .0004 = 24
   d. 1.704 × 2 = 3.408
   e. 3.1 × 400 = 124

8. From the pattern exhibited by the true sentences that follow:

   a. try to state a general rule about the effect of multiplying a whole number by a power of ten;
   b. try to state such a rule about the effect of multiplying a decimal by a power of ten; and
   c. relate (b) to the procedure for placing a decimal point in a product.

   736 × 10 = 7360
   736 × 100 = 73,600
   736 × 1000 = 736,000
   7.36 × 10 = 73.6
   7.36 × 100 = 736
   7.36 × 1000 = 7360
   7.36 × 10,000 = 73,600
   .00736 × 10,000 = 73.6
   .736 × 10 = 7.36

Division

Division of decimals is by far the most mysterious and troublesome of all the operations using decimals as far as justifying the rules and procedures used in the various algorithms for getting quotients goes. Again we do the operation as if only whole numbers were involved and rely on well known procedures to place the decimal point. But these procedures are tricky. Furthermore, new possibilities are open to us. For example, if the division doesn't "come out even," that is, if there is a remainder left, we can now add more zeros after the last decimal place in our divisor and go merrily on our way until either it does "come out even" or we stop for some other reason, e.g., boredom, instructions given us, or the conditions of the problem. If it still doesn't come out even, what do we now do with the "remainder"? Shall we "round off" or let it go? The discussion that follows will not deal with all the possible ramifications of these problems but will, hopefully, make clear why the principle maneuvers we use are sensible and justifiable.

First Method of Division:

From Chapter 9 we know how to do division problems and justify our results at least to the extent of getting a quotient and a remainder for any problem involving whole numbers. Chapter 22 points out briefly that any division problem using whole numbers will give a single rational number as quotient without any remainder. For example, $17 \div 4$ becomes $4\frac{1}{4}$ rather than a quotient of 4 and a remainder of 1 and the corresponding "check" becomes $17 = 4 \times 4\frac{1}{4}$ rather than $17 = 4 \times 4 + 1$. Hence, if we can convert our division of decimals problem into a division of whole numbers problem, it can certainly be handled using procedures already discussed. Of course, our quotient may involve fractions rather than decimals, but there is a way to change fractions to decimals which will be discussed near the end of this chapter so even this need not disturb us.

It remains, then, to show how a division of decimals problem can be changed to a division involving only whole numbers. The most efficient way to handle this is to use the fraction notation $a \div b = \frac{a}{b}$ to designate the division, carrying on the presumption introduced in Chapter 22 that such fractions behave in pretty much the same way as fractions involving only whole numbers and counting numbers. Now the examples in Figure 23-4 should make clear what our procedure is. The first example details what is going on by the use of fractions equivalent to our decimals, but the rest proceed directly without this step.

a. $2.5 \div .2 = \frac{2.5}{.2} = \frac{\frac{25}{10}}{\frac{2}{10}} \quad \frac{\frac{25}{10} \times \frac{10}{1}}{\frac{2}{10} \times \frac{10}{1}} = \frac{25}{2} = 25 \div 2 = 12\frac{1}{2}$ (or 12.5)

b. $.9 \div .7 = \frac{.9}{.7} = \frac{.9 \times 10}{.7 \times 10} = \frac{9}{7} = 9 \div 7 = 1\frac{2}{7}$

c. $53.75 \div .5 = \frac{53.75}{.5} = \frac{53.75 \times 100}{.5 \times 100} = \frac{5375}{50} = 5375 \div 50 = 107\frac{25}{50}$

$= 107\frac{1}{2}$ (or 107.5)

$$.5\overline{)53.75} \longrightarrow \begin{array}{r} 107 \\ 50\overline{)5375} \\ \underline{50} \\ 375 \\ \underline{350} \\ 25 \end{array} \longrightarrow 107\frac{25}{50}$$

d. $1.072 \div .4 = \frac{1.072}{.4} = \frac{1.072 \times 1000}{.4 \times 1000} = \frac{1072}{400} = 2\frac{272}{400} = 2\frac{68}{100}$ (or 2.68)

$$.4\overline{)1.072} \longrightarrow \begin{array}{r} 2 \\ 400\overline{)1072} \\ \underline{800} \\ 272 \end{array} \longrightarrow 2\frac{272}{400}$$

Figure 23-4. Examples of division using decimals.

Observe that the procedure is to multiply both dividend and divisor by a large enough power of ten so that both are whole numbers, then divide in the way usual for whole numbers. Observe also that in examples (a), (c) and (d) of Figure 23-4 the answer could easily be changed to an equivalent decimal answer. How the $1\frac{2}{7}$ in the example (b) could be changed to a decimal will be the subject of the last section of this chapter.

Second Method for Division:

The procedure just described is not the usual one, as you know. On the other hand we would probably, in teaching youngsters, arrive at our usual procedure for handling all problems efficiently only as the end result of a number of simpler special cases and less efficient but more easily explained procedures. In the end, however, we would typically go about a division of fractions problem as follows:

Example 1

$3.36 \div .8$

Procedure: (a) Move the decimal point in both divisor and dividend the same number of places so that the divisor (but not necessarily the dividend) is a whole number.

$$\underset{\curvearrowright}{.8}\,\overline{)3.\underset{\curvearrowright}{3}6}$$

(b) Do the problem as if it were division of whole numbers, i.e., ignore the decimal point.

$$\begin{array}{r} 4.2 \\ 8\overline{)33.6} \\ \underline{32} \\ 1\,6 \\ \underline{1\,6} \end{array}$$

(c) Place the decimal point in the quotient in such a way that the quotient has exactly the same number of decimal places as the revised dividend obtained by step (a) above.

$$\begin{array}{rl} 4.2 & \text{Tenths} \\ 8\overline{)33.6} & \text{Tenths} \end{array}$$

(d) If there is still a remainder when all the digits in the dividend have been used up, one can, if he likes, add more zeros in the dividend and continue the division process. The example at the right illustrates this. Since adding zeros increases the number of decimal places in the dividend, and since we insist that the quotient have exactly as many decimal places as the dividend, this automatically increases the number of decimal places in the quotient. In our example at right, the process just described gets an exact quotient very soon, but this does not always happen and one must decide when to stop and what to do with the last remainder. These last questions will not be dealt with here.

Example 2

$3.38 \div 8$

$$\begin{array}{r} 4\ 225 \\ \underset{\curvearrowright}{.8}\,\overline{)3.\underset{\curvearrowright}{3}\,800} \\ \underline{3\,2} \\ 1\,8 \\ \underline{1\,6} \\ 20 \\ \underline{16} \\ 40 \\ \underline{40} \end{array}$$

$$\begin{array}{rl} 4.225 & \text{Thousandths} \\ 8\overline{)33.800} & \text{Thousandths} \end{array}$$

Justification: Step (a) is most easily justified by the arguments used earlier. We write the division as a fraction, then multiply dividend and divisor by the power of ten which will make the divisor a whole number. In the present example,

$$3.36 \div .8 = \frac{3.36}{.8} = \frac{3.36 \times 10}{.8 \times 10} = \frac{33.6}{8} = 33.6 \div 8 .$$

Any number of examples will show that this has the effect of moving the decimal point the same number of places in both numbers.

We are clearly justified in adding as many zeros as we please after the final decimal place, as in step (d), for this is just a matter of using equivalent decimals, as was discussed early in this chapter.

The real problem, that of justifying the placement of the decimal point, is handled by remembering that each division of dividend by divisor must, by definition, give a quotient such that the multiplication of the quotient times the divisor must give the dividend as product. Since we have made the divisor a whole number, so that it has no decimal places, the number of decimal places in the quotient must be the same as in the dividend so that the whole number divisor times the decimal quotient will give exactly the same result as the decimal dividend. In the present case, starting with the revised problem with whole number divisor, $33.6 \div 8 = n$, means that $n \times 8 = 33.6$ and since only a whole number times a decimal to tenths gives tenths in the product, n must be a decimal expressed to tenths.

Problems

9. Rewrite each of the following numerals as a fraction with a counting number as denominator.

a. $\frac{73.6}{.25}$

b. $\frac{.097}{3.26}$

c. $\frac{685}{8.2}$

d. $\frac{350}{.007}$

e. $\frac{.649}{.36}$

10. If $\begin{array}{r} 154 \\ 15\overline{)2310} \end{array}$ find decimal numerals for the following without actually doing a full scale division.

a. 231.0 ÷ 15
b. 23.10 ÷ 15
c. 2.310 ÷ 15
d. 23.10 ÷ .15
e. .2310 ÷ 1.5
f. .02310 ÷ 15

Changing Fraction Names to Decimal Names

As we pointed out earlier, all the operations with decimals could have been done by changing the decimals to their fraction equivalents and using procedures already considered in some detail. In other words, we have just been doing things that raise no fundamentally new issues but are only alternates to known ways of proceeding. But, if we consider how to change a fraction name to an equivalent decimal name, it turns out that some really new issues are raised.

To go from a fraction name to a decimal name, we again identify $\frac{a}{c}$ as a ÷ c and perform a division. Some such conversions are shown in Figure 23-5.

a. $\frac{1}{2} = .5$ since

$$\begin{array}{r} .5 \\ 2\overline{)1.0} \\ \underline{1\ 0} \end{array}$$

b. $\frac{1}{25} = .04$ since

$$\begin{array}{r} .04 \\ 25\overline{)1.00} \\ \underline{1\ 00} \end{array}$$

c. $\frac{175}{10} = 17.5$ since

$$\begin{array}{r} 17.5 \\ 10\overline{)175.0} \\ \underline{10} \\ 75 \\ \underline{70} \\ 5\ 0 \\ \underline{5\ 0} \end{array}$$

Figure 23-5. Changing fractions to decimals.

In the cases shown in Figure 23-5 the division process terminates. For many fractions, however, the division process does not terminate. When we try it for $\frac{1}{3}$ or $\frac{3}{11}$ for example, we get $\frac{1}{3} = .333\ldots$ and $\frac{3}{11} = .2727\ldots$ as shown in Figure 23-6.

$\frac{1}{3} = .333\ldots$ since

$$\begin{array}{l} .333\ldots \\ 3\overline{)1.000\ldots} \\ \underline{9} \\ 10 \\ \underline{9} \\ 10 \\ \underline{9} \\ 1 \text{ and so on} \end{array}$$

$\frac{3}{11} = .2727\ldots$ since

$$\begin{array}{l} .2727\ldots \\ 11\overline{)3.0000\ldots} \\ \underline{2\ 2} \\ 80 \\ \underline{77} \\ 30 \\ \underline{22} \\ 80 \\ \underline{77} \\ 3 \text{ and so on} \end{array}$$

Figure 23-6. Non-terminating decimal equivalents for $\frac{1}{3}$ and $\frac{3}{11}$.

Observe, however, that although the division process does not terminate in the examples of Figure 23-6 a certain repeating pattern seems to occur. This will always happen for rational numbers.

Perhaps it would be instructive to show an expansion of $\frac{1}{3}$ using fractions.

$$\frac{1}{3} = \frac{10}{30} = \frac{9}{30} + \frac{1}{30} = \frac{3}{10} + \frac{1}{30}$$

In a similar way, $\frac{1}{30} = \frac{10}{300} = \frac{9}{300} + \frac{1}{300} = \frac{3}{100} + \frac{1}{300}$.

If the expansion of $\frac{1}{30}$ is combined with that for $\frac{1}{3}$ and the same process is continued we would get the expansion shown in Figure 23-7.

$$\begin{aligned}
\frac{1}{3} &= \frac{10}{30} = \frac{9}{30} + \frac{1}{30} = \frac{3}{10} + \frac{1}{30} \\
&= \frac{3}{10} + \left(\frac{10}{300}\right) = \frac{3}{10} + \left(\frac{9}{300} + \frac{1}{300}\right) = \frac{3}{10} + \left(\frac{3}{100} + \frac{1}{300}\right) \\
&= \frac{3}{10} + \frac{3}{100} + \left(\frac{10}{3000}\right) = \frac{3}{10} + \frac{3}{100} + \left(\frac{9}{3000} + \frac{1}{3000}\right) \\
&= \frac{3}{10} + \frac{3}{100} + \left(\frac{3}{1000} + \frac{1}{3000}\right) \\
&= \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{1}{3,000}
\end{aligned}$$

If this were continued indefinitely, we would get

$$\frac{1}{3} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10,000} + \frac{3}{100,000} + \ldots$$

or $$\frac{1}{3} = .3 + .03 + .003 + .0003 + .00003 + \ldots$$

or $$\frac{1}{3} = .33333\ \ldots$$

Figure 23-7. An expansion of $\frac{1}{3}$.

The examples that have been discussed seem to suggest that decimal expansions for rational numbers either terminate (like $\frac{1}{2} = 0.5$), or repeat (like $\frac{1}{3} = 0.333\ \ldots$). To check this, let us examine the divisior process we used.

Consider the rational number $\frac{7}{8}$. If we carry out the indicated division we would write:

$$\begin{array}{rl}
.875 & \\
8\overline{)7.000} & \\
\underline{6\;4} & \\
60 & \text{remainder } 6 \\
\underline{56} & \\
40 & \text{remainder } 4 \\
\underline{40} & \\
0 & \text{remainder } 0
\end{array}$$

In dividing by 8, the only remainders which can occur are 0, 1, 2, 3, 4, 5, 6 and 7. The only remainders which did occur were 6 then 4 and finally 0. When the remainder 0 occurs, the division is exact. Such a decimal is often spoken of as a terminating decimal.

What about a rational number which does not have a terminating decimal representation? Suppose we look again at $\frac{3}{11}$. The process of dividing 3 by 11 proceeded like this:

$$\begin{array}{rl}
.272 & \\
11\overline{)3.000} & \\
\underline{2\;2} & \\
80 & \text{remainder } 8 \\
\underline{77} & \\
30 & \text{remainder } 3 \\
\underline{22} & \\
8 & \text{remainder } 8
\end{array}$$

Here the possible remainders are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. Not all the remainders appear, but 8 and 3 do occur, and in this order. At the next stage in the division the remainder 3 recurs, so the sequence of remainders 8, 3 occurs again and again. The corresponding sequence of digits 2, 7 in the quotient will therefore occur periodically in the decimal expansion for $\frac{3}{11}$. This type of decimal is referred to as a repeating or periodic decimal.

In order to write such a periodic decimal concisely and without ambiguity, it is customary to write 0.2727 ... as $0.\overline{27}$.

The bar over the digit sequence 27 indicates the set of digits which repeats. Similarly, 0.333 ... is written as $0.\overline{3}$, $\frac{2}{7} = 0.\overline{285714}$, and $\frac{7}{30} = 0.2333 \ldots = 0.2\overline{3}$.

The method which has been discussed is quite a general one, and it can be applied to any fraction $\frac{a}{b}$. If the indicated division by $\underline{b}$ is performed, then the only possible remainders which can occur are 0, 1, 2, 3, ..., (b - 1). It is necessary to look at the stages which contribute to the digits that repeat in the quotient. These stages usually occur after zeros are annexed to a dividend to carry on a division that has not terminated.

Even a terminating decimal expansion like 0.25 may be written with a repeated zero as $0.25000 \ldots$ or $0.25\overline{0}$ to provide a periodic expansion. Note that a zero remainder may occur without terminating the division process, for example,

$$\begin{array}{rl}
905.3 & \\
6\,\overline{)\,5432.0} & \\
\underline{54} & \\
03 & \text{remainder } 0\\
\underline{0} & \\
32 & \text{remainder } 3\\
\underline{30} & \\
2\;0 & \text{remainder } 2\\
\underline{1\;8} & \\
2 & \text{remainder } 2
\end{array}$$

This shows that $\frac{5432}{6} = 905.\overline{3}$. If 0 does not occur as a remainder after zeros are annexed to the dividend, then after at most $(b - 1)$ steps in the division process one of the possible remainders $1, 2, \ldots, (b - 1)$ will recur and the digit sequence will start repeating.

We can see from this argument that any rational number has a decimal expansion which is periodic.

We have seen how to find by division the decimal expansion of a given rational number. But, suppose you have the opposite situation, that is, you are given a periodic decimal. Does such a decimal represent a rational number? The answer is that it does, and we show this in the following paragraphs. The demonstration is a bit tricky, however, and involves some algebraic techniques that may not be familiar to you so that you should feel perfectly free to skip it or read it only lightly.

This problem can be approached by considering an example. Let us write the number $0.2424\ldots$ and call it n, so that $n = 0.\overline{24}$. The periodic block of digits is 24. If you multiply by 100, this shifts the decimal point two places and gives the relation:

$$100 \times n = 100 \times .242424\ldots = 24.2424 \ldots .$$

Then, since

$$100 \times n = 24.2424\ldots$$

and

$$n = 0.2424$$

you can subtract n from each side of the first equation to yield:

$$99n = 24 \quad \text{so that,}$$

$$n = \frac{24}{99} \quad \text{or, in simplest form,}$$

$$n = \frac{8}{33}\,.$$

You find by this process that $0.\overline{24} = \frac{8}{33}$.

The example here illustrates a general procedure which mathematicians have developed to show that any periodic decimal represents a rational number. You see, therefore, that there is a one-to-one correspondence between the set of rational numbers and the set of periodic decimals. It would be quite possible then for us to define the rational numbers as the set of numbers represented by all such periodic decimals. A question naturally arises about non-periodic decimals. What are they? Certainly not rational numbers. The fact that such non-periodic decimals exist should suggest to us that perhaps there are numbers which are not rational numbers. We will discuss this situation at some length in Chapter 30.

Computing with non-terminating decimals presents many problems, as you can easily verify by attempting, say, $.333\ldots \times .2727\ldots$. A brief consideration of such issues is also included in Chapter 30.

Summary

In this chapter we have discussed decimals as an equivalent way of naming rational numbers and as a natural extension of our place value system for writing numerals. Procedures used for computing sums, differences, products and quotients using decimals were discussed. It was noted that it is simple indeed to write fractions equivalent to given terminating decimals but not always such an easy matter to find decimal equivalents for given fractions. We found that such a decimal equivalent for a given fraction might not terminate, but that even if it does not, a repeating pattern of digits becomes evident. Furthermore, any decimal with such a repeating pattern of digits does represent a rational number and its fraction equivalent can be found, though this is sometimes hard to do.

Exercises - Chapter 23

1. Tell the number represented by each 3. Tell the number represented by each 5.
   - a. 321.59
   - b. 71.035
   - c. 5421.365
   - d. 720.513
   - e. 49.035
   - f. 795.309

2. Express the following numbers as decimals.
   - a. $\frac{3000}{8}$
   - b. $\frac{300}{8}$
   - c. $\frac{30}{8}$
   - d. $\frac{3}{8}$
   - e. $\frac{3}{80}$
   - f. $\frac{3}{800}$

3. Find decimal names for these quotients and "check" by multiplication.
   - a. 1008 ÷ .6
   - b. 213.9 ÷ 3.75
   - c. 646 ÷ 6.8
   - d. 30.94 ÷ 2.6

4. Make the following sentences true by supplying missing decimal points or zeros in dividend, divisor, or quotient as required. (It is true that 8153 ÷ 263 = 31.)
   - a. 8.153 ÷ 263 = 3.1
   - b. 8.153 ÷ 26.3 = 31
   - c. 8153 ÷ 26.3 = 3.1

5. If 78 × 75 = 5850, supply a missing decimal point and/or zeros in either factor or in the product as required to make each a true sentence.
   - a. .0078 × 75 = .05850
   - b. 78 × 7.5 = 58.50
   - c. 78. × 7.5 = 5850
   - d. .075 × 78 = 585.0

6. Write a decimal numeral for $\frac{1}{13}$.
   - a. Does this decimal end?
   - b. How soon can you recognize a pattern?
   - c. What is the set of digits which repeats periodically?

7. Write the decimals for:

   a. $\frac{2}{3}$

   b. $\frac{7}{8}$

   c. $\frac{1}{9}$

   See how soon you can recognize a pattern in each case. In performing the division, watch the remainders. They may give you a clue about when to expect the decimal numeral to begin to repeat.

8. Write the decimals for:

   a. $\frac{1}{11}$ b. $\frac{4}{11}$ c. $\frac{14}{11}$ (There is a "shortcut" for doing (b) and (c).)

9. Is it true that the number $0.\overline{63}$ is seven times the number $0.\overline{09}$?

10. Find the decimal numeral for the first number in each group. Then calculate the others without dividing.

    a. $\frac{1}{5}$, $\frac{2}{5}$, $\frac{4}{5}$

    b. $\frac{1}{20}$, $\frac{3}{20}$, $\frac{11}{20}$

    c. $\frac{1}{1000}$, $\frac{111}{1000}$, $\frac{927}{1000}$

*11. Find fraction names for these rational numbers:

    a. $0.\overline{36}$

    b. $0.\overline{142857}$

    c. $0.\overline{9}$

12. Express the answer to each of the following as a decimal numeral.

    a. An automobile used 10.5 gallons of gasoline in travelling 163.8 miles. How many miles per gallon is this?

    b. How long would it take to travel 144 miles at 50 miles per hour?

    c. One day Helen and Rosemary were each given a guinea pig. Helen's guinea pig weighed 0.60 pounds and gained 0.07 pounds each day. Rosemary's guinea pig weighed 0.48 pounds, but ate more, and gained 0.09 pounds each day. Whose guinea pig was the heavier a week later? How much heavier?

    d. In a swimming test, Dan stayed under water 2.3 times as long as Charlie. Charlie stayed under water 19.8 seconds. How long did Dan stay under water?

    e. Races are sometimes measured in meters. If a meter is 1.094 yards, what is the difference in yards between a 50 meter race and a 100 meter race?

Solutions for Problems

1. a. 9876.543 b. 702.907

2. a. $(9 \times 100) + (2 \times 10) + (7 \times 1) + (8 \times \frac{1}{10}) + (7 \times \frac{1}{100}) + (2 \times \frac{1}{1000})$

   b. $(4 \times 10) + (9 \times \frac{1}{100})$

   or, if you like, $(4 \times 10) + (0 \times 1) + (0 \times \frac{1}{10}) + (9 \times \frac{1}{100})$.

   c. $(4 \times 1) + (6 \times \frac{1}{100{,}000})$, or

   $(4 \times 1) + (0 \times \frac{1}{10}) + (0 \times \frac{1}{100}) + (0 \times \frac{1}{1000}) + (0 \times \frac{1}{10{,}000}) + (6 \times \frac{1}{100{,}000})$

3. a. 400.061; Possible errors: 400.060; .461 ("Four hundred sixty-one thousandths"); .460 ("Four hundred sixty one-thousandths").

   b. 2,300.0040 Possible error: .2340

4. a. > b. > c. = d. >

5. 0.02, .25, 1, 1.02, 2.002, 2.2, 2.25

6. a.

$$\begin{array}{r} 8.900 \\ 3.000 \\ \underline{5.375} \\ 17.275 \end{array} \qquad \begin{array}{rcl} 8\frac{9}{10} & = & 8\frac{900}{1000} \\ 3 & = & 3 \\ 5\frac{375}{1000} & = & \underline{5\frac{375}{1000}} \\ & & 16\frac{1275}{1000} = 17\frac{275}{1000} \end{array}$$

$$8.9 + 3 + 5.375 = \frac{89}{10} + \frac{3}{1} + \frac{5375}{1000} = \frac{8900}{1000} + \frac{3000}{1000} + \frac{5375}{1000} = \frac{17{,}275}{1000}$$

b.

$$\begin{array}{r} 40.00 \\ \underline{-12.57} \\ 27.43 \end{array} \qquad \begin{array}{rcl} 40 & = & 39\frac{100}{100} \\ 12\frac{57}{100} & = & \underline{12\frac{57}{100}} \\ & & 27\frac{43}{100} \end{array}$$

$$40 - 12.57 = \frac{40}{1} - \frac{1257}{100} = \frac{4000}{100} - \frac{1257}{100} = \frac{2743}{100}$$

7. a. $.7 \times .9 = .63$

   b. $.7 \times .02 = .014$

   c. $.006 \times .0004 = .0000024$

   d. Nothing required

   e. $3.1 \times 400 = 1240.0$

8. a. Multiplying a whole number by 10 or 100 or 1000, etc., has the effect of making the product have the same digits as the other factor but with one zero or two zeros or three zeros, etc., "tacked" on at the end.

   b. Multiplying a decimal by a power of 10 has the effect of moving the decimal point to the right one place for multiplier 10, two places for multiplier 100, three places for multiplier 1000, etc.

   c. This latter is because in multiplying a decimal by a power of ten as if the decimals didn't exist we "tack on" zeros. Then the product must have as many decimal places as the decimal factor (since the powers of ten are whole numbers) and in counting these decimal places off in the product, the "tacked on" zeros are counted. Hence, the decimal place ends up just as many places to the right of its former position as there are zeros tacked on; e.g., one place for 10, two places for 100, etc.

9. a. $\frac{7360}{25}$ c. $\frac{6850}{82}$ e. $\frac{64.9}{36}$

   b. $\frac{9.7}{326}$ d. $\frac{350,000}{7}$

10. a. 15.4 d. 154

    b. 1.54 e. .154

    c. .154 f. .00154

Chapter 24

RATIO, RATE, PERCENT

In our study of whole numbers and of rational numbers, we have always considered a physical situation, first looking at its characteristic qualities and properties. We have then tried to extract from this look at the physical world the ideas and properties of number which are basic to the study of mathematics.

Thus we looked at the way in which certain sets of objects were alike and developed the concept of whole numbers. We know that a set of 5 apples and that a second set of 5 letters of the alphabet can be put in a one-to-one correspondence. These sets have something in common. We denote the fundamental property in which we are interested by the number 5. By considering joins of sets and arrays of sets, we had physical models of the ideas of addition and multiplication.

We represented whole numbers by points on a number line and found that further consideration of other points on a number line gave us a good physical model of the rational numbers. A study of this and other physical models helped us understand the addition and multiplication properties of rational numbers.

In a similar way, physical models such as paper triangles, lines drawn on a chalkboard, solid boxes, etc., helped us in the study of the mathematical concepts of points, lines, planes, curves and other geometrical figures. We will later study congruence and similarity, concepts which grow out of our desire to compare models of geometric figures as to size and shape.

We come now to another similar physical situation whose study will give us a different look at numbers and show how they can be useful in a new and interesting way. The concept of ratio, which we will develop in this chapter, will give us still one more way of using numbers to indicate how certain physical situations are alike.

Consider the following problem. John can buy 2 candy bars for 6¢, while Jim can get 6 of the same candy bars for 20¢. They wonder who is getting the better "buy." It is assumed that there is no special discount for large purchases. Since John knows that he must present 6¢ for every two candy bars, he can visualize his candy purchasing ability as pictured below. Every 2 candy bars must correspond to 6 pennies.

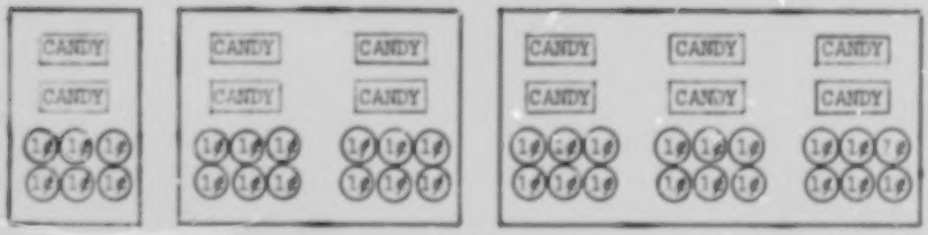


Figure 24-1.

The last frame clearly indicates that John is doing better than Jim is doing under these arrangements, for he is paying 18¢ for six candy bars while Jim is paying 20¢ for 6 candy bars.

Exactly how did we reach this conclusion? At first we asked ourselves what sort of purchase would be like the purchase of 2 candy bars for 6¢. The situations represented above are a partial answer. To sharpen our understanding of how these situations are alike, let us summarize the essentials of each situation in a table.

CANDY BARS	2	4	6	8	10
PENNIES	6	12	18		

Figure 24-2.

Can we make further entries in our table? If we are able to visualize or draw a picture of the situation, we can make the corresponding table entry with ease.

We notice that an essential aspect of each situation we have described can be represented by using a pair of numerals: (2, 6) for the first frame, (4, 12) for the second, and (6, 18) for the third. These pairs can be used to represent a property common to all of these situations. Using the pair (2, 6) we introduce the symbol 2:6 (read 2 to 6). In terms of the above model, this can be interpreted as telling us that there are 2 candy bars for every 6 pennies. This same correspondence could have been described using the pair (4, 12) and the associated symbol 4:12. For the above model this would tell us that there are 4 candy bars for every set of 12 pennies. Clearly, 4:12 and 2:6 are different symbols which we can use to indicate the same kind of correspondence and we write 2:6 = 4:12. Once more, as in the case of numerals for rational numbers, we have an unlimited choice of symbols to represent the same property. The common property is called a ratio. In the preceding example, the ratio of candy bars to pennies is said to be 2 to 6 or 4 to 12. An alternate way of expressing the same ratio is to write $\frac{2}{6} = \frac{4}{12}$. Such forms behave much as rational numbers do. The way to tell if two ratios written this way are equal is to take the cross product. This procedure was explained in Chapter 19.

$$\frac{a}{b} = \frac{c}{d} \text{ if } a \times d = c \times b$$

$$\frac{2}{6} = \frac{4}{12} \text{ if } 2 \times 12 = 4 \times 6$$

Can we tell how much John will have to pay for one candy bar? If we take another look at the first frame of Figure 24-1, we see that two candy bars cost six cents. By rearranging this frame as in Figure 24-3, it becomes apparent that one candy bar should cost three cents.

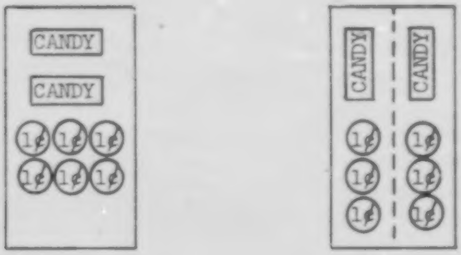


Figure 24-3. 2:6 = 1:3.

How much candy can I buy for 1¢? In trying to answer this question, we find ourselves incapable of describing the situation by a suitable pair of numbers, unless we consider the candy bars to be divisible. In fact, the candy bars are divisible although the store owner is not likely to sell us part of a candy bar. If he would, we would expect to get $\frac{1}{3}$ of a candy bar for a penny. Hence, $\frac{1}{3}:1$ is another name for the ratio we have been studying. However, if the candy store owner won't cut up the candy bar, this particular pair of numbers doesn't describe a situation that will actually occur at the candy store.

We have seen that the ratios 1:3, 2:6, 4:12, 6:18, $\frac{1}{3}:1$ can all be used to describe the basic property that each element of the first set, the set of candy bars, always corresponds to 3 elements of the second set, the set of pennies. We might now ask, can we without drawing pictures, decide which pairs of numbers can be used to describe this ratio? Clearly any pair of the form $(n, 3 \times n)$ where $\underline{n}$ is a rational number will do. Of course, if the storekeeper will not subdivide the candy bars and if the penny is the smallest unit of money available, only pairs of the form $(n, 3 \times n)$ where $\underline{n}$ is a whole number will represent actual transactions at the candy counter. That is, while $5:15$ and $\frac{1}{5}:\frac{3}{5}$ both represent the same ratio, we see that $5:15$ tells us that 5 candy bars will cost us 15¢, whereas $\frac{1}{5}:\frac{3}{5}$ tells us that $\frac{1}{5}$ candy bar would cost $\frac{3}{5}$ ¢. Since the dealer will not sell us $\frac{1}{5}$ candy bar, $\frac{1}{5}:\frac{3}{5}$ does not actually describe a possible exchange of money for candy bars as $5:15$ does.

The property described by $2:6$ is exhibited in a wide variety of situations and is not restricted to sets of candy bars and pennies. Consider each of the following:

1) There are 2 texts for every 6 students.
2) There are 2 boys for every 6 girls in class.
3) The motor scooter does 2 miles in 6 minutes.
4) My investment earns $2.00 interest for every $6.00 invested.

After a brief consideration, you will conclude that the table and the associated pictures which we developed for our example of candy bars and pennies would serve equally well to describe each of the above situations. For example, in (1) instead of candy bars, we have texts and instead of pennies, we have students.

Consider the statement (1). It describes a situation involving 2 sets: a set of texts and a set of students. The situation in question exhibits a property described by $2:6$. We can say that the ratio of number of texts to number of students is 2 to 6. In short, there are 2 texts for every 6 students. Another name for this ratio is $3:9$. This indicates that there are 3 texts for every 9 students. $1:3$ also describes the ratio of the number of texts to the number of students. However, the ratio of the number of students to the number of texts is $3:1$, i.e., 3 members of the set of students correspond to each member of the set of texts. Clearly, in making comparisons between numbers of texts and numbers of students it will not do to say that the ratio is $1:3$, unless we understand that the first number indicated refers to the set of texts. The order in which the numbers are named is important. Any pair of the form $(n, 3 \times n)$ when interpreted as $n:3 \times n$ could be used to describe the relationship between the set of texts and the set of students. That is, since there are $\underline{n}$

texts for every $3 \times n$ students, we have a situation exhibiting the ratio property 1:3. Some pairs of this type are given in the following table. Spaces are provided for further entries.

TEXTS	1	3	5	12
STUDENTS	3	9	15	36

Figure 24 4.

We can, of course, never hope to list all possible entries. We can visualize what the table entries tell us about our model sets as shown below.

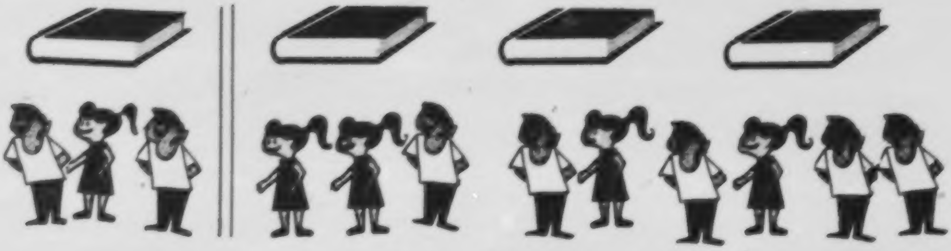


Figure 24-5.

We might ask how to spot one of our table entries without drawing a picture. In the examples we've been considering, the pairs we enter in our table are all of the type $(n, 3 \times n)$ and we see at once that $(9, 27)$ will represent a table entry, while $(4, 17)$ will not.

Consider the ratio described by 2:3. This symbol tells us that there are 2 items of the first set for every 3 items of the second. It follows that 4:6, 6:9, 100:150, and, in general, $2 \times k:3 \times k$ would be other ways of representing this same property. If the first set referred to is the set of boys in school and the second set is the set of girls, we say that there are 2 boys for every three girls in school. The symbol 2:3 can also be used to describe a fundamental aspect of what happens when we have a motorscooter which travels at the rate of 2 miles every 3 minutes. In other words, corresponding to every 2 mile stretch covered by the motorscooter, there is a time interval of 3 minutes. 2:3 will describe a correspondence exhibited here if we choose the elements of the first set to be distances of one mile and the elements of the second to be time intervals of one minute. The aspect of the movement of the motorscooter is equally well described by any symbol of the form $2 \times k : 3 \times k$. Some such pairs are indicated below.

MILES	2	4	5	10	30	$\frac{1}{2}$	1	$\frac{2}{3}$
MINUTES	3	6	$\frac{15}{2}$	15	45	$\frac{3}{4}$	$\frac{3}{2}$	1

Figure 24-6.

Situations in which the correspondence of two sets can be described as above, by means of pairs of numerals of the type $(a \times k, b \times k)$ or $a \times k:b \times k$ all possess a property called the ratio $a:b$. That is, to each collection of a members of the first set there corresponds a collection of b members of the second. If two pairs of numerals represent the same ratio, we use an equal sign to show that they are different names for the same property. For example, $5:10 = 4:8$. A statement of this type is called a proportion.

How can we tell if two symbols, for example $6:18$ and $8:32$, represent the same ratio? The symbol $6:18$ tells us that there are 6 members of the first set for every 18 members of the second. This is the same as 1 member of the first set for every 3 members of the second set. That is, $6:18$ and $1:3$ are different names for the same ratio. Similarly, $8:32$ and $1:4$ are different names for the same ratio. These symbols, $1:3$ and $1:4$, clearly describe different correspondences and we conclude that $6:18$ and $8:32$ do not represent the same ratio. In general, $a:b$ $(a \neq 0)$ and $(b \neq 0)$ represents the same ratio as $1:\frac{b}{a}$ while $c:d$ represents the same ratio as $1:\frac{d}{c}$ $(c \neq 0)$ and $(d \neq 0)$. It follows that $a:b$ and $c:d$ can represent the same ratio if and only if $\frac{b}{a} = \frac{d}{c}$. That is, $a:b = c:d$ if and only if $a \times d = b \times c$. Using this test, we see immediately that $6:18 \neq 8:32$, for $6 \times 32 \neq 18 \times 8$.

When we wish to compare physical objects of the same kind, we use numbers to measure their size. In the case of line segments, we assign numbers to measure their length. Thus we can compare the physical objects by considering the ratio of their measures. If two line segments, $\overline{AB}$ and $\overline{CD}$, have lengths 5 and 3 centimeters respectively, we can compare their size by saying their measures are in ratio five to three. We write this in the proportion $m(\overline{AB}):m(\overline{CD}) = 5:3$. ($m(\overline{AB})$ stands for the measure of $\overline{AB}$.)

Thus, if a desk measures 24 inches by 30 inches, we can write $\ell:w = 30:24 = 5:4$.

A special kind of ratio or rate is that of percent. Here 100 is always the basis for comparison. In fact, percent means "per hundred." Thus 25% means 25 per hundred. When written as a ratio, this would be 25:100 or $\frac{25}{100}$ or .25. It may again be rewritten as $\frac{1}{4}$, $\frac{2}{8}$ or as many other equivalent fractions as you please. From this we see that percent can be treated as a special type of ratio which can be converted to equivalent fraction forms and decimal forms.

In general any number $\frac{a}{b}$ *can be expressed as a percent by finding the number* c *such that*

$$\frac{a}{b} = \frac{c}{100} .$$

By studying this pattern, we see that if we are given any two of the three numbers a, b, c we can find the third. Thus, since $\frac{3}{4} = \frac{75}{100}$, *b* is 4 in $\frac{3}{b} = \frac{75}{100}$; *a* is 3 in $\frac{a}{4} = \frac{75}{100}$; *c* is 75 in $\frac{3}{4} = \frac{c}{100}$.

Consider a situation in which over a fixed period of time I can earn \$1.50 on a \$50 investment. From what I know about simple interest, I would expect to get \$0.75 on a \$25 investment, \$0.03 on a \$1 investment, etc. If, as before, we use a table to exhibit these results, we would have:

DOLLARS OF INTEREST	0.30	1.50	0.75	3	0.03
DOLLARS INVESTED	10	50	25	100	1

A property common to all of these pairs is the ratio 0.03:1. In particular, note the pair (3, 100). This can be interpreted to tell us that we receive \$3.00 of interest for every \$100 invested. If we use this pair to describe the ratio property, we write 3:100 and indicate that we get a return of 3 *per* 100 or 3 *percent*. We use the symbol 3% (read 3 per cent) to describe how our interest compares dollar for dollar with our investment.

The following problems illustrate some applications of percent.

1. A florist has fifty rose bushes and sells 12 of them. What percent does he sell? The solution to this problem is: 12 is what percent of 50? We can form a ratio of 12:50 or $\frac{12}{50}$ and convert if to a percent by writing it in an equivalent form: $\frac{12}{50} = \frac{24}{100}$. Thus our answer is 24%. Checking with our formula, $\frac{a}{b} = \frac{c}{100}$, we replace *a*

by 12 and b by 50 to get $\frac{12}{50} = \frac{c}{100}$. The answer, of course, is 24.

2. In another instance we may wish to find the answer to the problem: what is 20% of 80? Again, using the formula, we see that $\frac{a}{80} = \frac{20}{100}$ and the answer is 16.

3. Twenty-four students received a passing grade in Professor A's history examination. He announced that 80% of his class passed the examination. How many students were in the class?

   In this instance the formula holds, $\frac{a}{b} = \frac{c}{100}$. Therefore $\frac{24}{b} = \frac{80}{100}$ and the answer is 30.

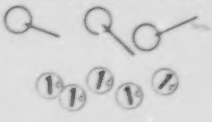
In studying correspondences between two sets, we were led to the concept of ratio. Think about the following statements and you should begin to appreciate the wide applicability of this new idea.

Problem*

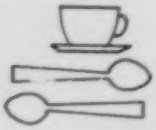
1. Express the following statements as ratios.
   a. The population is 200 people per square mile.
   b. The car travelled 100 yards in 6 seconds.
   c. The recipe calls for 3 cups of sugar for every cup of water.
   d. The scale on this floor plan is $1\frac{1}{4}$ centimeters per 10 feet.
   e. I can buy 2 sweaters for 7.
   f. My investment is earning 4% interest.

Exercises - Chapter 24

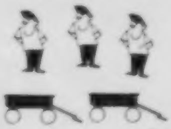
1. Study these pictures. Write a symbol which describes the comparison.



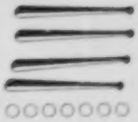
(a)



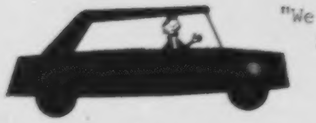
(b)



(c)



(d)


"We've gone 8 miles in 10 minutes."

(e)

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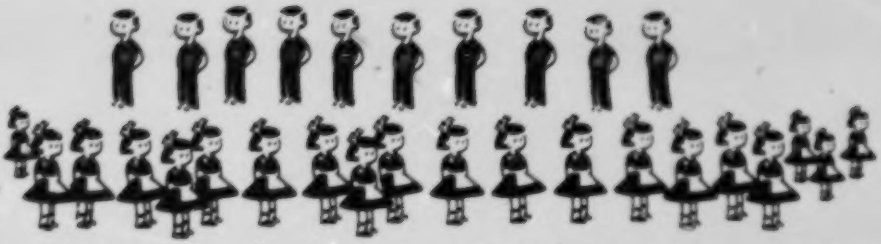
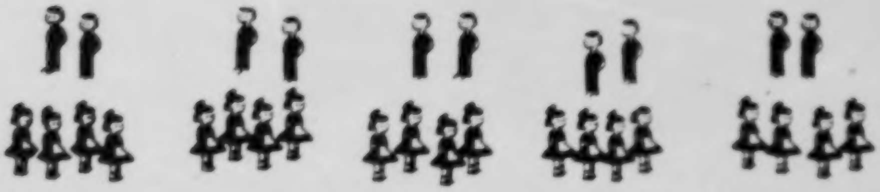
* Solution to the problem in this chapter is on page 325.

2. Look at this picture of a fifth grade class.



a. What is one way of writing the symbol which represents the ratio of boys to girls?



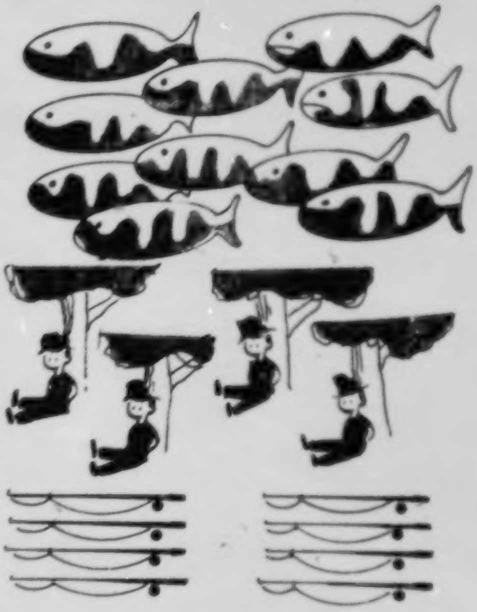
Write a symbol to express this ratio of boys to girls.

3. a. Write two symbols which express the ratio of the number of fish to the number of boys.

b. Write two symbols which express the ratio of the number of boys to the number of fish.

c. Write two symbols which describe the ratio of the number of boys to the number of fishpoles.

d. Write two symbols which describe the ratio of fishpoles to boys.



4. Copy and complete this table.

4:1	16:__	8:__	20:__	36:__	__:25	12:__	__:6	__:8	__:10	1:__

5. Copy and complete each of these three tables.

(a)	(b)	(c)
4:8	10:4	6:10
1:__	30:__	__:20
8:__	5:__	__:30
__:32	__:8	30:__
__:4	__:20	24:__
12:__	__:24	__:80
__:48	100:__	__:100
__:40	40:__	54:__
__:72	__:32	36:__
32:__	1,000:__	42:__
3:__	15:__	3:__
__:1	__:10	__:15

6. One day a sixth-grade pupil heard the principal say, "Four percent of the fifth graders are absent today." A list of absentees for that day had 22 names of fifth-grade pupils on it. From these two pieces of information, the sixth-grade pupil discovered the number of fifth-grade pupils in the school. How many fifth-grade pupils are there?

7. It is often more convenient to refer to the data at some later time if they are given in percent than if they are given otherwise.

For example: the director of a camp left some records for future use. Some information was given as percent, and some was not. The records gave the following items of information.

a. There were 200 boys in camp.

b. One hundred percent of the boys were hungry for the first dinner in camp.

c. On the second day in camp 44 boys caught fish.

d. One boy wanted to go home the first night.

e. A neighboring camp director said, "Forty percent of the boys in my camp will learn to swim this summer. We shall teach 32 boys to swim."

From (a) and (b), how many hungry boys came to dinner the first day?
From (c) find the percent of boys who caught fish the second day.
From (d) find the percent of the boys who were homesick.
From (e) find the total number of boys in the second camp.

Solutions for Problems

1. a. 200:1
   b. 100:6 or 50:3
   c. 3:1
   d. $\frac{5}{4}$:10 or 5:40 or 1:8
   e. 2:7
   f. 4:100 or 1:25

Chapter 25

CONGRUENCES AND SIMILARITIES

Introduction

We return now for a second look (see Chapters 13-16) at some ideas of geometry. In Chapter 13 we considered a few properties of points, lines and planes and their relationships. In Chapter 14 simple closed curves were discussed briefly, and particular attention was drawn to certain special simple closed curves such as triangles, polygons and circles and the regions bounded by them. In Chapter 15 we talked about congruence of line segments and of angles. In this chapter we shall extend the notion of congruence from congruence of line segments and angles to congruence of triangles. What does congruence mean for triangles? Under what conditions are two triangles congruent? Having looked at congruence of triangles, which has to do with their size and shape, we shall discuss briefly the relationship of similarity between triangles. This relationship concerns the shape of geometric figures without reference to their size.

Congruence

Congruence may be defined as follows:

> Two geometric figures which have the same size and shape are said to be congruent.

This is not a technical definition of congruence but it tells us what we want to know. The figures below are congruent in pairs.

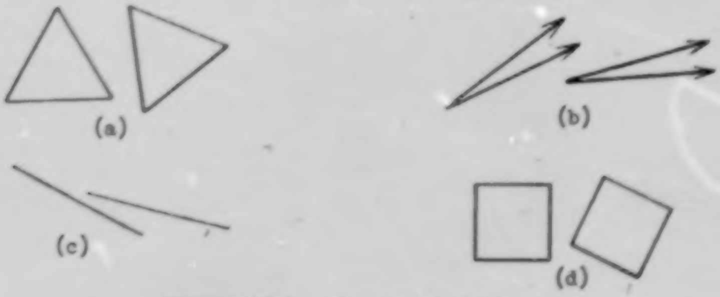


Figure 25-1. Congruent figures.

We observe in passing that even though congruence of line segments and angles was defined before their measurement was discussed, it is surely true that if two segments are congruent, they have the same measure, provided they are measured in terms of the same unit. The same is true of angles. Thus in Figure 25-2, $\overline{RS}$ and $\overline{PQ}$ are congruent and also have the same length. $\angle B \cong \angle C$ and these two angles have the same measure. (Note that, if there is no ambiguity possible, we abbreviate the name of the angle by naming just the vertex point instead of naming also points on each ray of the angle).

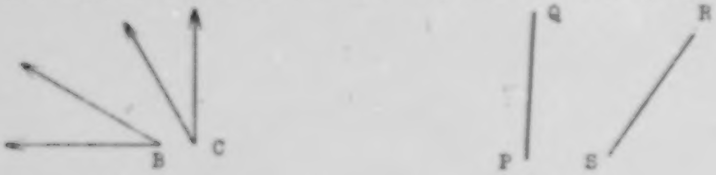


Figure 25-2. $\angle B \cong \angle C$; $\overline{PQ} \cong \overline{RS}$

We know that congruence of segments and angles can be determined by direct comparisons of representations of the figures. The same is true of any two plane figures. If we think that two circles might be congruent, or two triangles or two quadrilaterals, etc., we could always make a model of one and try to match it with the other. If we can match them we know the original figures are congruent. But can congruence of two plane figures be established in any other manner? Does it follow from the congruence of all the pairs of angles and segments determined by the figures? Yes, it does, but a more interesting question is: how little do we have to know to be sure two figures are congruent? Are two triangles congruent if their pairs of sides are congruent? If their pairs of angles are congruent? We will see later that the answer to the first question about triangles is yes while that to the second is no.

Are two circles congruent if they have congruent radii? Yes. In Figure 25-3, $\overline{OP} \cong \overline{QR}$, and it can be verified in the usual way that these two circles are congruent.

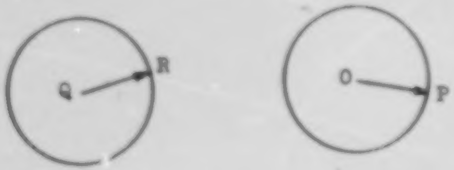


Figure 25-3. $\overline{OP} \cong \overline{QR}$, circle $O \cong$ circle Q.

It is a fact that:

> Two circles are always congruent if their radii are congruent.

Problems*

1. Make a tracing of circle O and check that circle O is congruent to circle Q.
2. Draw an arbitrary segment $\overline{AB}$. Draw a segment $\overline{RS} \cong \overline{AB}$. Draw circles with centers A and R whose radii are congruent to $\overline{AB}$ and $\overline{RS}$. Check that these circles are congruent.

Are two rectangles congruent if their bases are congruent? No, because they may have different heights. But if their heights and bases are congruent, Figure 25-4 makes it appear that the rectangles will also be congruent.

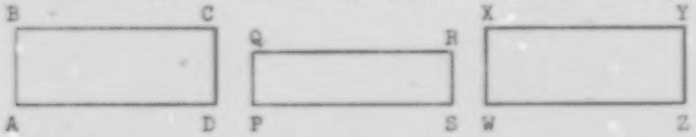


Figure 25-4. $\overline{AD} \cong \overline{PS} \cong \overline{WZ}$, $\overline{AB} \cong \overline{XW}$, $\overline{AB} > \overline{PQ}$.
Rectangle ABCD $\cong$ Rectangle WXYZ.
Rectangle ABCD $\ncong$ Rectangle PQRS.

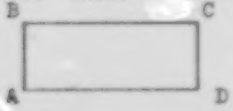
In fact it is true that:

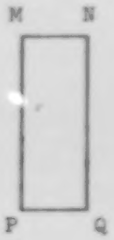
> Two rectangles are congruent if their bases and heights are respectively congruent.

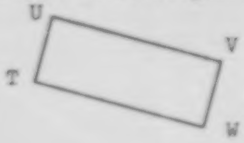
Thus two conditions are necessary for the congruence of rectangles while one condition was enough for congruence of circles.

Problems

3. Make a tracing of ABCD in Figure 25-4 and check that it is congruent to WXYZ.
4.







Check that rectangle ABCD $\cong$ rectangle MNPQ $\cong$ rectangle TUVW. Change the statement about congruent rectangles given above to take care of this new situation.

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* Solutions for problems in this chapter are on page 339.

Congruence of Triangles

In Figure 25-5, the three triangles are congruent.

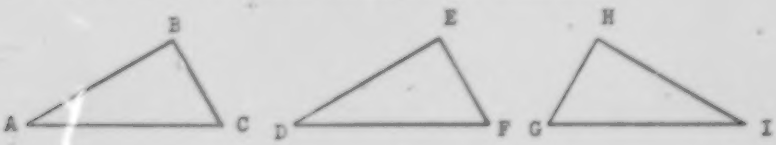


Figure 25-5. Congruent triangles.

If $\triangle DEF$ were traced on paper and the paper cut along the sides of the triangle, the paper model would represent a triangle and its interior. In this discussion we are interested only in the triangle and not in its interior. The paper model of $\triangle DEF$ could be placed on $\triangle ABC$ and would fit exactly. If point D were placed on point A with $\overline{DF}$ along $\overline{AC}$, point F would fall on point C, and point E would fall on point B. In these two triangles there would be six pairs of congruent segments and congruent angles which may be displayed as follows:

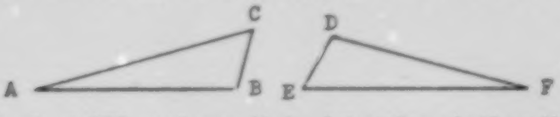
$$\overline{AB} \cong \overline{DE} \qquad \angle B \cong \angle E$$
$$\overline{AC} \cong \overline{DF} \qquad \angle A \cong \angle D$$
$$\overline{CB} \cong \overline{FE} \qquad \angle C \cong \angle F$$

In this case, we say that triangle ABC is congruent to triangle DEF and write $\triangle ABC \cong \triangle DEF$, being very careful to name the triangles in such a fashion that corresponding letters are names for matching points. Thus, in this case it would be incorrect to say $\triangle ABC \cong \triangle FED$ since in the congruence of angles $\angle A \cong \angle D$ and not to $\angle F$. For these congruent triangles then, for each angle or side of one triangle there is a congruent angle or side in the other triangle.

If the paper model of $\triangle DEF$ is placed on $\triangle GHI$ it would match only if point D matches point I and points E and F match points H and G respectively. Note that to obtain this matching the model will have to be turned over. In this case $\triangle DEF \cong \triangle IHG$ with $\overline{DE} \cong \overline{IH}$ and $\angle E \cong \angle H$, etc. Do you see why it would be incorrect to say that $\triangle DEF \cong \triangle GHI$?

Problem

5.



Make a model of $\triangle ABC$ and use it to determine whether the two triangles are congruent. If they are, write seven congruence statements which are true, three for angles, three for segments and one for triangles.

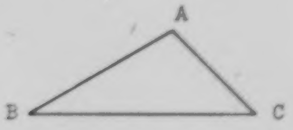
Making models of several different pairs of congruent triangles will make the following statement plausible. As a matter of fact it is true.

> If two triangles are congruent then the three sides of one are congruent respectively to the three sides of the other and the three angles of the one are congruent respectively to the three angles of the other.

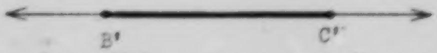
The pairs of congruent sides or angles are called corresponding pairs of sides or angles of the two triangles. Thus in Figure 25.5 $\overline{AB}$ and $\overline{DE}$ are corresponding segments and $\angle C$ corresponds to $\angle F$.

The statement above says that the congruence of two triangles gives us information about the congruence of the pairs of corresponding sides and angles. Let us turn the situation around and investigate the question: "How much must be known about the congruence of the sides and angles of two triangles to be sure that the triangles are congruent?" If all six pairs of corresponding sides and angles are congruent, the triangles are congruent. But perhaps three pairs will be enough. If so will it be any three pairs or only certain sets of three? In the next few paragraphs it will be shown that there are several different sets of three pairs of corresponding parts, such as the three pairs of sides, which are enough. It will also be shown that there are several different sets, such as the three pairs of angles, which are not enough to make the triangles congruent.

We will try to answer the question by making some simple experiments with physical models. Suppose we were asked to draw a triangle which is congruent to $\triangle ABC$.



1. Start by laying off a segment $\overline{B'C'}$ congruent to $\overline{BC}$. (Note: B', read "B prime," is a symbol used to represent a point corresponding to B.)



Then take a line segment $\overline{B'D}$ congruent to $\overline{BA}$ and using it as a radius, draw a circle with center B'. Next take a segment $\overline{C'E}$ congruent to $\overline{CA}$ and, using it as a radius, draw a circle with center C'. These two circles will intersect at two points. Label one of them R and draw $\overline{B'R}$ and $\overline{C'R}$.

Now, by the properties of congruence for segments and the definition of a circle, $\overline{B'R} \cong \overline{BA}$ and $\overline{C'R} \cong \overline{CA}$, and we started out by drawing $\overline{B'C'} \cong \overline{BC}$. You will find on checking that $\triangle RB'C'$ is congruent to $\triangle ABC$. See Figure 25-6.

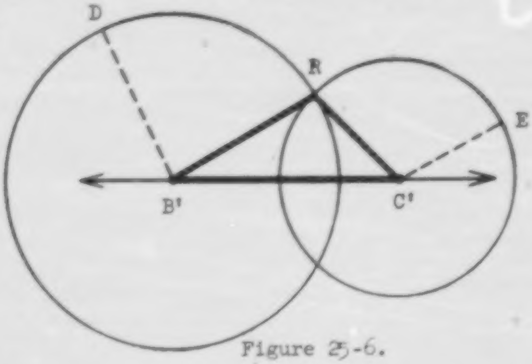


Figure 25-6.

Thus in this case it seems true that by copying three sides of $\triangle ABC$ we were able to draw a triangle congruent to it.

2. Another method of drawing such a triangle might be to start again by laying off $\overline{B'C'} \cong \overline{BC}$. Next we might try to draw an angle at B' congruent to B.

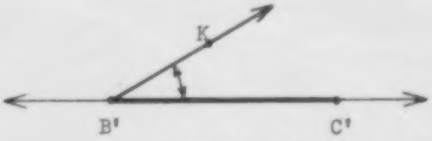


Figure 25-7.

An easy way to do this is to trace a copy of $\angle B$ at point B'. Figure 25-7 shows $\overrightarrow{B'K}$ drawn so that $\angle B' \cong \angle B$. As a next step there are two possibilities to be considered.

a. Mark A' on $\overrightarrow{B'K}$ so that $\overline{B'A'} \cong \overline{BA}$ and draw $\overline{A'C'}$ thus getting $\triangle A'B'C'$. See Figure 25-8.

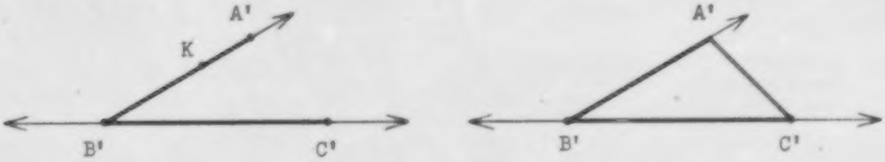


Figure 25-8.

b. Draw $\overrightarrow{C'L}$ so that $\angle B'C'L \cong \angle C$. Then $\overrightarrow{B'K}$ and $\overrightarrow{C'L}$ will intersect at a point which is labeled A" (read A double prime). This gives us $\triangle A''B'C'$. See Figure 25-9.

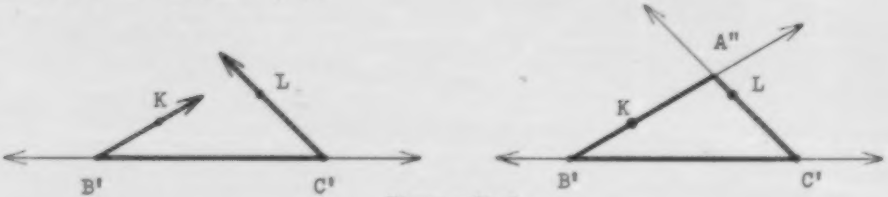


Figure 25-9.

We now have $\triangle A'B'C'$ and $\triangle A''B'C'$. In both cases only three parts of $\triangle ABC$ have been copied.

If your drawings and measurements are correct you will find that:

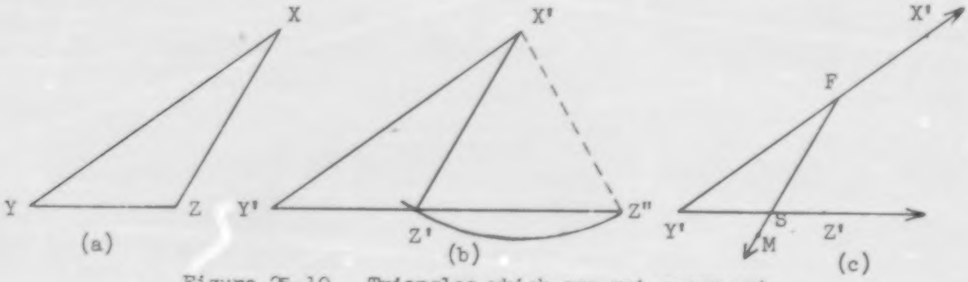
$$\triangle ABC \cong \triangle A'B'C' \quad \text{and} \quad \triangle ABC \cong \triangle A''B'C'.$$

In method (a) by copying one angle, $\angle B$, of $\triangle ABC$ and laying off on its sides segments congruent to $\overline{BC}$ and $\overline{BA}$ we have been able to draw a triangle congruent to $\triangle ABC$.

In method (b) by copying one segment, $\overline{BC}$, of $\triangle ABC$ and at its endpoints, copying the two angles $\angle B$ and $\angle C$, we have been able again to draw a triangle congruent to $\triangle ABC$.

In these two cases as in the first one it seems true that by copying three particular parts of $\triangle ABC$ we were able to draw a triangle congruent to $\triangle ABC$.

These are essentially the only cases which produce congruence in the two triangles. Suppose we try to copy $\triangle XYZ$ in Figure 25-10a by copying $\angle XYZ$ at Y' and marking off $\overline{Y'X'} \cong \overline{YX}$. If we try to mark off $\overline{X'Z'} \cong \overline{XZ}$, the result may give us $\triangle X'Y'Z''$ instead of $\triangle X'Y'Z'$ and $\triangle X'Y'Z''$ is certainly not congruent to $\triangle XYZ$. Likewise if, in Figure 25-10c, we copy $\angle X$ not at X', but at some arbitrary point on Y'X', say F, $\overrightarrow{FM}$ and $\overrightarrow{Y'Z'}$ will meet at some point S but $\triangle FY'S$ will not be congruent to



(a) (b) (c)

Figure 25-10. Triangles which are not congruent.

$\triangle XYZ$. It is true, however, that $\angle FSY' \cong \angle XZY$.

Thus we see that even if the three pairs of angles of two triangles are congruent, the triangles may not be. Nor need the triangles be congruent if 2 pairs of sides and one pair of angles are congruent. Careful repetition of drawings in the other three cases should make it convincing that:

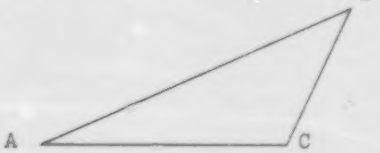
Two triangles are congruent if:

1. Three sides of one triangle are congruent respectively to three sides of the other triangle.
2. Two sides and the angle which lies between them of one triangle are congruent respectively to two sides and the angle which lies between them of the other triangle.
3. Two angles and the side which lies between them of one triangle are congruent respectively to two angles and the side which lies between them of the other triangle.

These are in fact properties of triangles which we will accept on the basis of our experiments.

Problems

6.



Use the first method of copying the three segments of $\triangle ABC$ to draw a triangle $A'B'C'$. Then by tracing make a model of $\triangle ABC$ and see if $\triangle A'B'C' \cong \triangle ABC$.

7. Use the second method and draw $\triangle PQR$ by copying $\overline{AC}$, $\angle ACB$ and $\overline{CB}$ of $\triangle ABC$ in Problem 6. Check that $\triangle PQR \cong \triangle ABC$.

8. Use the third method and draw $\triangle XYZ$ by copying $\overline{AC}$, $\angle BAC$ and $\angle BCA$ of $\triangle ABC$ in Problem 6. Check.

9. Make tracings of $\triangle PQR$ and $\triangle XYZ$ and see if they are congruent to $\triangle A'B'C'$.

Similarity of Triangles

While discussing congruence of triangles we found a situation as shown in Figure 25-10c where two triangles which have three angles of one triangle congruent respectively to three angles of the other are not necessarily congruent. There is, however, a very definite relationship between the two triangles. They look alike even though they are not the same size.

We said at the beginning of this chapter that we were going to call such figures similar. More formally:

> Two geometric figures which have the same shape though not necessarily the same size are said to be similar.

As with our definition of congruence, this is not a complete technical definition, but it is good enough for us now. The conclusion we were led to for triangles can be stated:

> Two triangles which have three angles of one congruent to three angles of the other are similar.

Is there anything we can say about the corresponding sides of similar triangles?

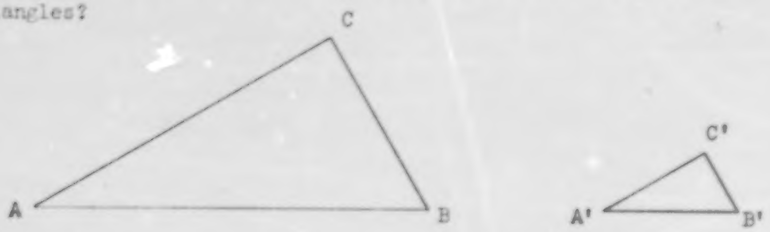


Figure 25-11. Similar triangles.

Suppose we measure the sides of $\triangle ABC$ and $\triangle A'B'C'$ where we know that $\angle A \cong \angle A'$, $\angle B \cong \angle B'$ and $\angle C \cong \angle C'$. We find that if in terms of a certain unit the measures of $\overline{A'B'}$, $\overline{B'C'}$ and $\overline{A'C'}$ are the numbers 4, 3 and 2, then in terms of the same unit the measures of $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$ are respectively 12, 9 and 6. Thus, in this case the ratio of the length of $\overline{AB}$ to the length of $\overline{A'B'}$ is $12:4 = 3:1$. As in Chapter 16, we use the symbol $m(\overline{AB})$ to represent the measure (length) of $\overline{AB}$. We have then as in Chapter 24:

$m(\overline{AB}) : m(\overline{A'B'}) = 12 : 4 = 3 : 1$. Also we have
$m(\overline{BC}) : m(\overline{B'C'}) = 9 : 3 = 3 : 1$ and
$m(\overline{AC}) : m(\overline{A'C'}) = 6 : 2 = 3 : 1$.

But what we have found for this triangle will prove to be true for any two similar triangles. We have then the general statement:

> If two triangles are similar, then the measures of their corresponding sides always have the same ratio.

This is true not only for triangles but for any pair of geometric figures which are similar to each other. This is the mathematics behind architects' drawings of building plans, road maps, scale models, etc. In such situations the "scale" of the drawing or model is usually given so you can figure the actual size of an object by using the measurements in the drawing of the object along with the given "scale" ratio. Thus, if a house plan is scaled $\frac{1}{4}$ inch to 1 foot, a room whose plan is 6 by 8 inches will measure 24 by 32 feet.

A warning. We found that two triangles would be similar if the three angles of one are congruent to the three angles of the other. This is not true however for other polygons. Thus, it is obvious that the two rectangles in Figure 25-12 have congruent angles since all their angles are right angles. It is also obvious that the rectangles do not have the same shape, and that the ratios of the lengths of corresponding sides are not the same.

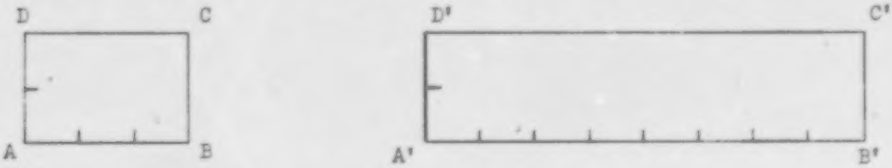
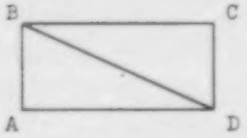


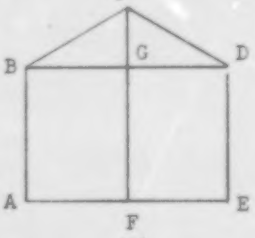
Figure 25-12. $m(\overline{AB}) : m(\overline{A'B'}) = 3 : 8$ but $m(\overline{AD}) : m(\overline{A'D'}) = 2 : 2$.

Exercises - Chapter 25

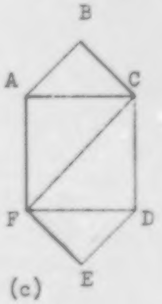
1. By tracing triangles on a sheet of thin paper find the triangles in each part which are congruent to each other. Be sure to name corresponding vertices in order. In part (a) state your answer like this: $\triangle BAD \cong \triangle DCB$. In parts b, ..., f you may have to trace more than one triangle.



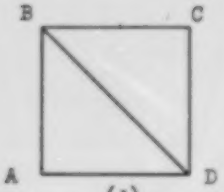
(a)



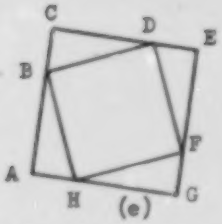
(b)



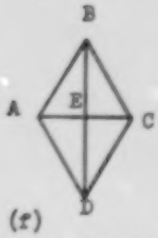
(c)



(d)

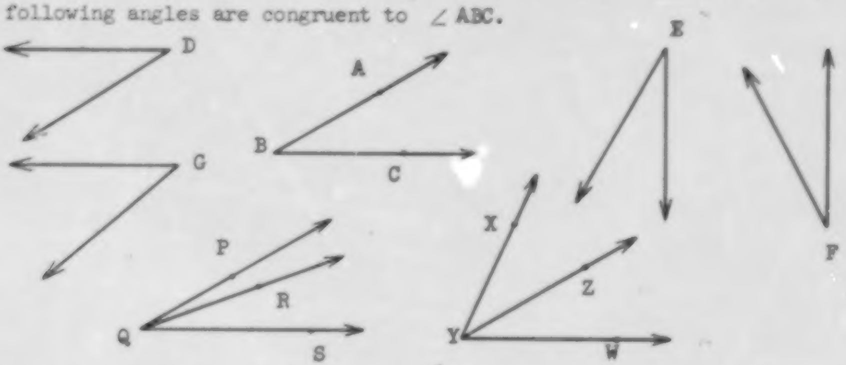


(e)



(f)

2. By tracing $\angle ABC$ on a sheet of thin paper, determine which of the following angles are congruent to $\angle ABC$.



3. $\triangle ABC \cong \triangle PQR$. Write the six correct congruences for the sides and angles of the two triangles.

4. If in $\triangle XYZ$ and $\triangle LMN$ we know that $\overline{XY} \cong \overline{MN}$, $\overline{YZ} \cong \overline{NL}$ and $\angle XYZ \cong \angle MNL$, do we know that the triangles are congruent? If so, write the correct statement of congruency.

5. Suppose we know that in $\triangle ABC$ and $\triangle DEF$, $\angle BAC \cong \angle EFD$, $\overline{AB} \cong \overline{EF}$ and $\overline{BC} \cong \overline{ED}$. Are the triangles congruent? If so write the correct statement of congruency.

6. $\triangle RST \cong \triangle PQR$. $m(\overline{RS}) = 5$, $m(\angle RST) = 42$. What can be said about the sides or angles of $\triangle PQR$?

7. (a) If $\triangle ABC \cong \triangle A'B'C'$, are the triangles similar?
   (b) If $\triangle ABC$ is similar to $\triangle A'B'C'$, must the triangles be congruent?

8. In each of the following, $\triangle ABC$ and $\triangle A'B'C'$ are two similar triangles, in which A and A', B and B', C and C' are pairs of corresponding vertices. Fill in the blanks where it is possible. Where it is not possible, explain why.

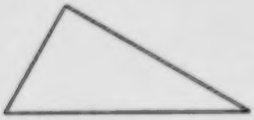
(a) $m(\angle A) = 30$, $m(\angle B) = 75$, $m(\angle A') = ?$, $m(\angle B') = ?$

(b) $m(\overline{AB}) = 3$, $m(\overline{AC}) = 4$, $m(\overline{A'B'}) = 6$, $m(\overline{A'C'}) = ?$

(c) $\frac{m(\overline{AB})}{m(\overline{A'B'})} = \frac{2}{5}$, $m(\overline{BC}) = 6$, $m(\overline{B'C'}) = ?$ $m(\overline{A'B'}) = ?$

(d) $\frac{m(\overline{BC})}{m(\overline{B'C'})} = \frac{4}{7}$, $m(\overline{A'C'}) = 14$, $m(\overline{AC}) = ?$ $m(\overline{BC}) = ?$

9. Find pairs of figures which are congruent.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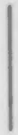
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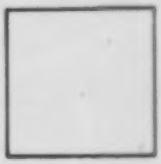
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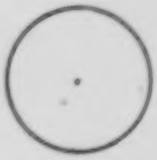
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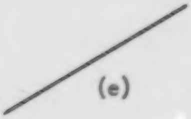
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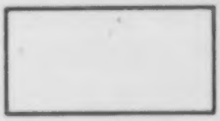
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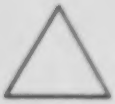
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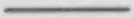
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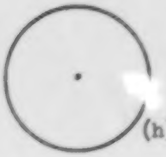
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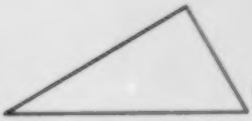
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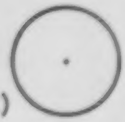
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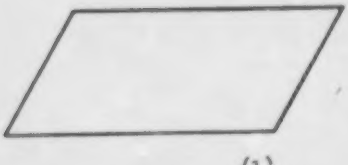
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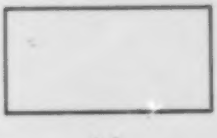
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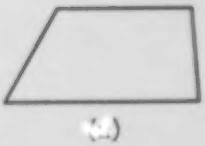
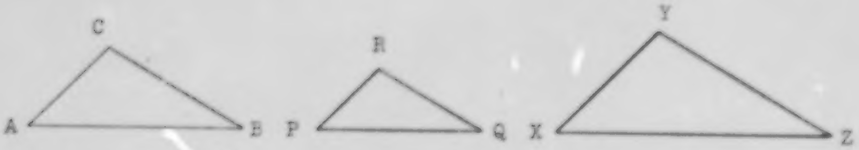

(n)

10.



These three triangles are similar, and $m(\overline{AB}) : m(\overline{PQ}) = 4 : 3$ while $m(\overline{AB}) : m(\overline{XZ}) = 4 : 5$. If in terms of a certain unit $m(\overline{AB}) = 12$, what are $m(\overline{PQ})$ and $m(\overline{XZ})$? If $m(\overline{PR}) = 15$ what is $m(\overline{AC})$ and what is $m(\overline{XZ})$?

11. In an architect's drawing, the scale is given as $\frac{1}{4}$ " to 1'. How big should the plan of a room be if the room is to measure 17' by 23'?

12. (a) The scale of a map is $\frac{1}{4}$ " to 10 miles. If the distance on the map from city A to city B is $2\frac{1}{8}$ " what is the actual distance between the cities?

(b) On the map of part (a), a salesman who lives at A finds the distance from A to B is $2\frac{1}{8}$ ", B to C is $3\frac{3}{8}$ ", from C to D is $4\frac{1}{2}$ " and from D to A is $1\frac{1}{16}$ ". How far does he travel to visit all four cities and return home?

Solutions for Problems

1,2,3. You should check the statements with models.

4. The new statement might be:

> Two rectangles are congruent if each of a pair of adjacent sides of one is congruent respectively to one of a pair of adjacent sides of the other.

5. $\triangle ABC \cong \triangle FDE$, $\overline{AC} \cong \overline{FE}$, $\overline{BC} \cong \overline{DE}$, $\overline{AB} \cong \overline{FD}$, $\angle ABC \cong \angle FDE$, $\angle ACB \cong \angle FED$, $\angle BAC \cong \angle DFE$.

6,7,8,9. Check with models.

Chapter 26
SOLID FIGURES

Introduction

We started the consideration of geometry in this course with a discussion of points, lines and planes. Their properties and intersection possibilities were studied. Next simple closed curves were taken up, but we soon confined our attention to curves which lay in a plane and considered such plane figures as triangles, rectangles and circles.

Now we are going to take a look at some of the more common solid figures, those which do not lie in a plane. Our pictures of them, of course, will be in a plane--the plane of the sheet of paper you are reading--and some people find it hard to visualize a solid figure from a picture of it. We will try to draw careful figures which may help you. It would also be well for you to try to draw some of these pictures yourself. This may enable you to visualize the solid figures we are talking about. An even greater help would be to procure or make actual models of the figures we are talking about.

Pyramids

Consider the triangle ABC. It lies in a certain plane. The union of the triangle and its interior is a special case of a plane region which we call a triangular region. Select a point D which is not in the same plane as $\triangle ABC$.

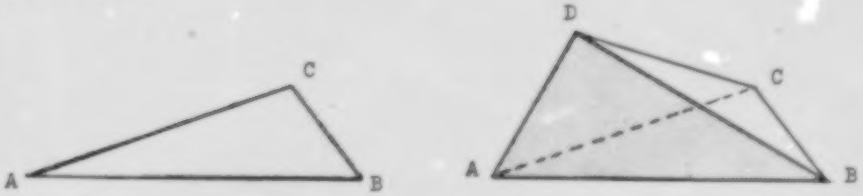


Figure 26-1.

The line segments which can be drawn from D to points in $\overline{AB}$ all lie in the triangular region DAB. Likewise, those from D to points in $\overline{AC}$ and from D to points in $\overline{BC}$ respectively, lie in the triangular regions DAC and DBC. These four triangular regions ABC, DAB, DBC and DAC form a pyramid whose base is the triangular region ABC, whose vertices are A, B, C and D, whose lateral faces are DAB, DAC and DBC and whose edges $\overline{AB}$, $\overline{BC}$, $\overline{CA}$, $\overline{DA}$, $\overline{DB}$ and $\overline{DC}$ outline the triangular regions. This particular pyramid is an example of the special class of pyramids called triangular pyramids, because its base is a triangular region. Any other polygon such as the quadrilateral ABCD, or the pentagon PQRST in Figure 26-2 may determine the base of a pyramid.

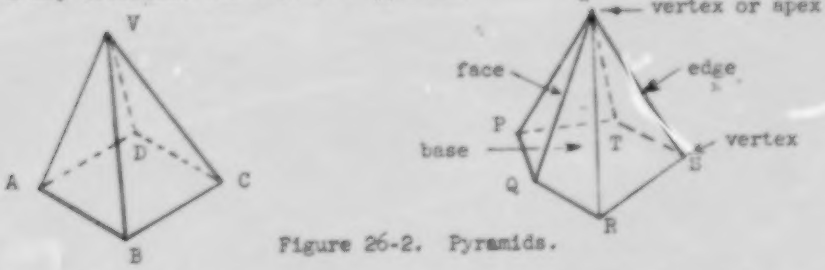


Figure 26-2. Pyramids.

It should be noted that while the base of a pyramid may be any polygonal region, each lateral face is always triangular. Each region is called a face of the pyramid. The intersection of any two faces is a segment called an edge and the intersection of any three or more edges is a point called a vertex. The pyramid is the union of all the faces. If you think of a solid model of the situation the pyramid is the surface of the solid and not the solid itself. The distinction is much the same as the one we made before between "triangle" and "triangular region." We have:

> A pyramid is a surface which is a set of points consisting of a polygonal region called the base, a point called the apex not in the same plane as the base, and all the triangular regions determined by the apex and each side of the base.

A pyramid is an example of a simple closed surface. There are many other simple closed surfaces which compare to our pyramid somewhat as a simple closed curve does to a triangle. We will consider some of the others such as prisms, cylinders and spheres in this unit.

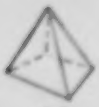
A characteristic property of a simple closed curve in a plane was that it separated the points of the plane other than those of the curve itself into two sets, those interior to the curve and those exterior to it. In the same manner a simple closed surface divides space, other than the set of points on its surface, into two sets of points, the set of points interior to the simple closed surface and the set of points exterior to the simple closed surface. One must pass through the simple closed surface to get from an interior point to an exterior point.

In a plane, we called the union of a simple closed curve and the points in its interior a plane region. In a similar manner, we will call the union of a simple closed surface and the points in its interior a solid region.

The pyramid is the surface of the solid region which it encloses. A pyramid is classified as triangular, quadrangular, pentagonal, etc., depending on whether the polygon outlining the base is a triangle, quadrilateral, pentagon, etc. See Figures 26-1 and 26-2. Although the word "pyramid" technically refers to the surface of a certain solid, it is frequently used outside of mathematics and sometimes in mathematics to refer to the solid. For instance, "The Pyramids of Egypt," mean the actual stone structures and not just their surfaces. Usually the context makes clear the meaning intended.

Problems*

1. Which of the following are drawings of pyramids?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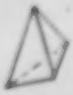
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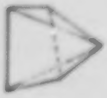
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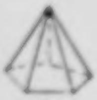
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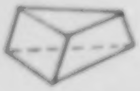
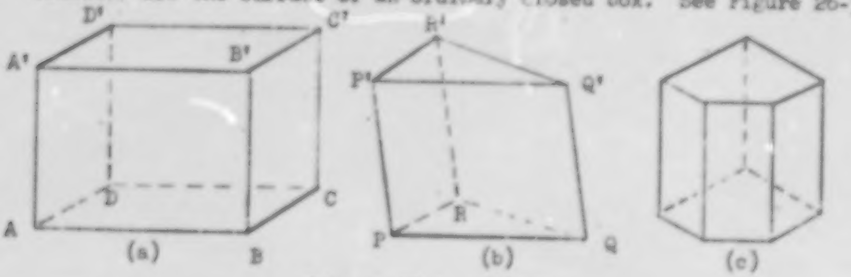

(e)

2. Indicate the base, a face different from the base, a vertex and an edge for each pyramid.

Prisms

Consider now the surface of an ordinary closed box. See Figure 26-3a.



(a) (b) (c)

Figure 26-3. Prisms.

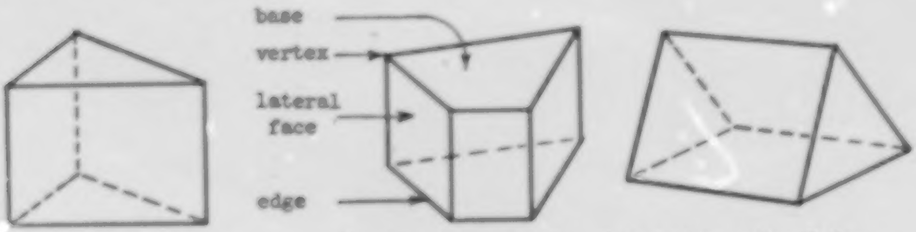
This is a special case of a surface called a prism. The bases of this prism are the rectangles ABCD and A'B'C'D' which lie in parallel planes and which are congruent. The edges $\overline{AA'}$, $\overline{BB'}$, $\overline{CC'}$, etc. whose endpoints are the corresponding vertices of the bases are all parallel to each other. They determine the lateral faces ABB'A', BCC'B', etc. Figure 26-3b shows another surface whose bases PQR and P'Q'R' are parallel and lie in parallel planes. Again the edges $\overline{PP'}$, $\overline{QQ'}$ and $\overline{RR'}$ are all parallel to each other. This surface is also an example of a prism, as is that in Figure 26-3c. In (a) and (c) the lateral faces are rectangles, but in (b) they are only parallelograms. These examples lead to the general definition of a prism.

---

* Solutions for problems in this chapter are on page 353.

> A prism is a surface consisting of the following set of *points*: two polygonal regions bounded by congruent polygons which lie in parallel planes; and a number of other plane regions bounded by the parallelograms which are determined by the corresponding sides of the bases.

Each of the plane regions is called a *face* of the prism. The two faces formed by the parallel planes mentioned in the definition are called the *bases* and the other faces are called *lateral faces*. The intersection of two adjacent faces of a prism is a line segment, called an *edge*. The intersections of lateral faces are called lateral edges. Each endpoint of an edge is called a *vertex*. Some further examples of prisms are illustrated in Figure 26-4.



Triangular prism

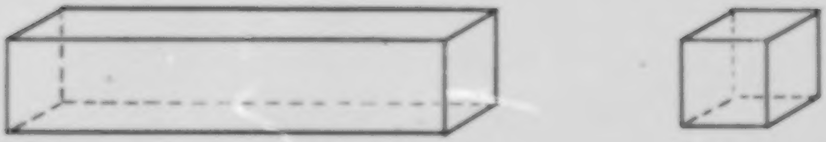
Quadrangular prism

Triangular prism lying on a lateral face

Figure 26-4.

All of the prisms in Figure 26-4 are examples of a special type of prism called a *right prism* in which the lateral edges are perpendicular to the base. Hence all the lateral faces are rectangles. A prism which is not a right prism is shown in Figure 26-3b.

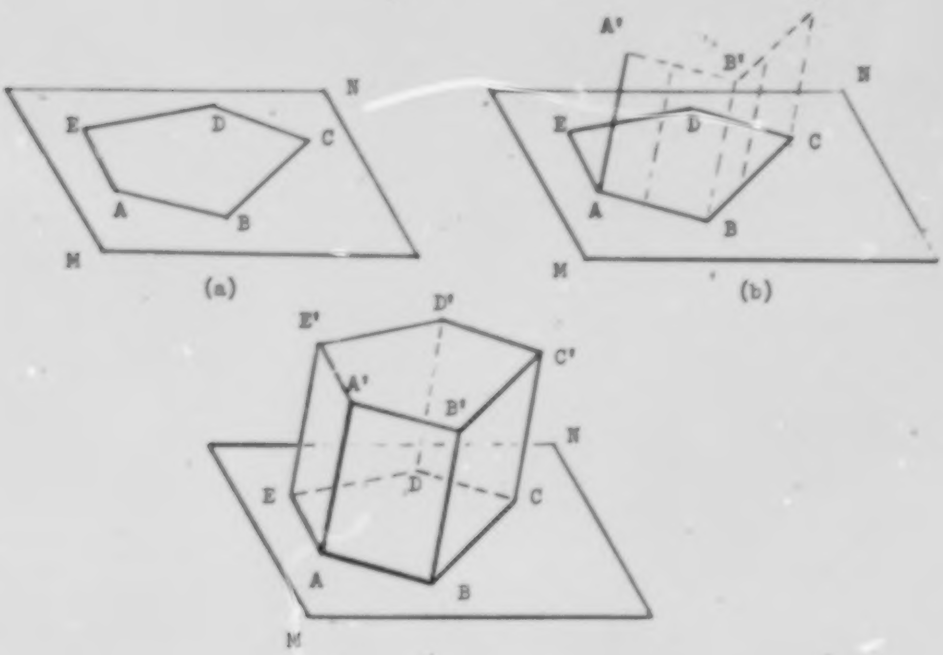
If the polygon outlining the base is a triangle, the prism is called a *triangular prism*. A prism is a quadrangular prism if the polygon is a quadrilateral, and pentagonal if the polygon is a pentagon. The special quadrangular right prism in which the quadrilateral is a rectangle is called a rectangular prism. If the base is a square and *each* lateral face is also a square, we get the familiar *cube*. These last two are shown in Figure 26-5.



A rectangular prism cube

Figure 26-5.

Another way to think of a prism is this. Consider any polygon such as ABCDE in Figure 26-6a which lies in the horizontal plane MN. Take a pencil to represent a line segment $\overline{PQ}$ and put one end of it at A. Move the pencil along $\overline{AE}$ keeping it always parallel to the original position as in (b). Then move it along $\overline{BC}$ still parallel to the original position and so on around the polygon. The pencil itself determines a surface and the upper tip of it outlines a polygon A'B'C'D'E' congruent to ABCDE. These two polygonal regions and the surface determined by the moving pencil make up the prism.

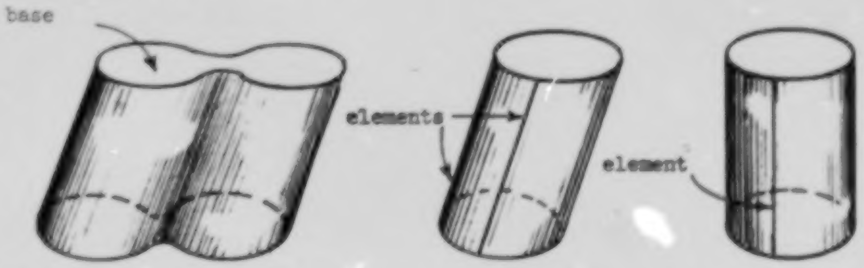


(a) (b)

Figure 26-6. A new look at a prism.

Cylinders

A cylinder is defined in a manner very similar to the way we defined a prism, except that the bases are regions bounded by congruent simple closed curves instead of polygons. Thus, in a very general sense, the prism is just a special case of a cylinder. A line which connects two corresponding points in the curves bounding the bases is called an element of the cylinder.



(a) (b) (c)

Figure 26-7. Cylinders.

In Figures 26-7b and 26-7c, the simple closed curve is a circle and the cylinder is called a circular cylinder. If an element is perpendicular to the plane containing the curve, we get a right cylinder. Common examples are, of course, a tin can or a hat box. A can of beans is a good model of a right circular cylinder while a can of sardines is a good model of a right cylinder which is not usually circular.

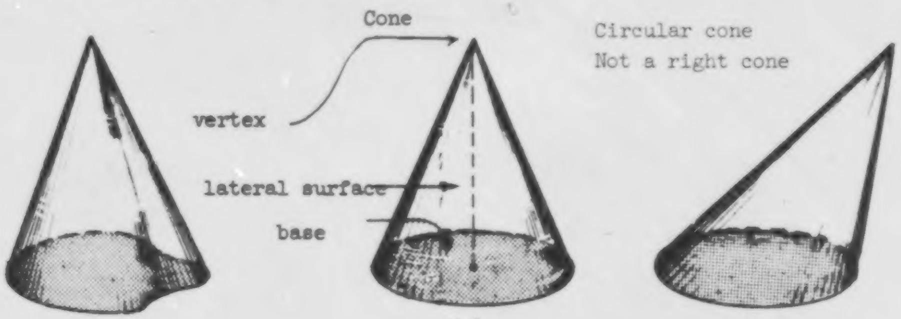
Cones

A cone is related to a cylinder as a pyramid is to a prism.

> A cone is a surface which is a set of points consisting of a plane region bounded by a simple closed curve, a point called the vertex not in the plane of the curve and all the line segments of which one endpoint is the vertex and the other, any point in the given curve.

This differs from the definition of a pyramid essentially only in the fact that we have changed "polygon" to "simple closed curve." Thus in a very general sense, a pyramid is a special type of cone.

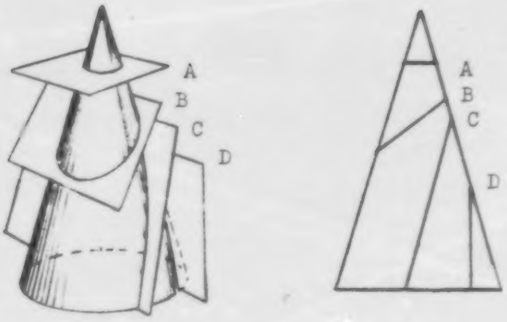
If the simple closed curve is a circle, we get a circular cone. A cone has a base, a lateral surface and a vertex.



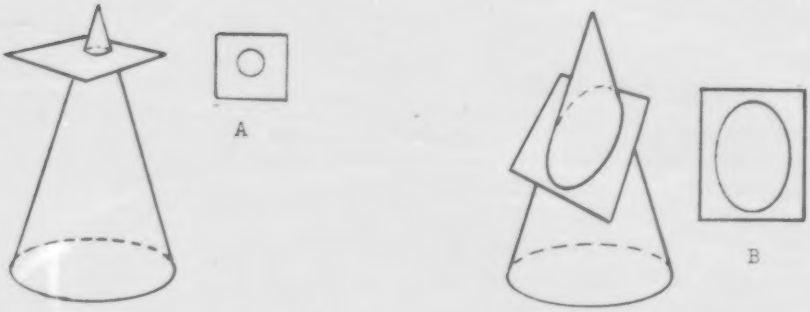
(a) (b) (c)

Figure 26-8. Cones.

The most familiar cone is the one illustrated in Figure 26-8b whose base is bounded by a circle and in which a line drawn from the vertex to the center of the base is perpendicular to the plane in which the base lies. This kind of a cone has been studied a great deal from ancient times to the present. If we made a model of wood or plastic of the solid region bounded by such a cone, we can cut the model by planes in several different directions with interesting results in each case. See Figure 26-9. A cut by a plane parallel to the base gives rise to a circle while other planes give rise to other interesting curves known collectively as the conic sections.

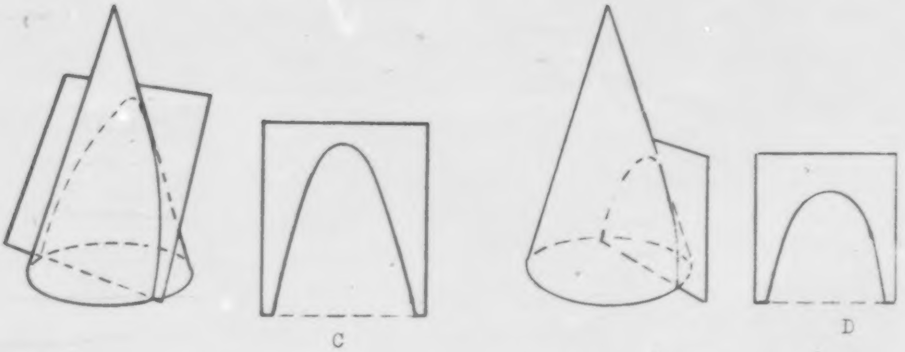


(a) Different planes cutting a cone.



(b) Circle: cut made by plane A parallel to base of cone.

(c) Ellipse: cut made by plane B not perpendicular to base.



(d) Part of parabola. Cut made by plane C parallel to element.

(e) Part of hyperbola: cut made by plane D perpendicular to base.

Figure 26-9. Sections of a cone.

Spheres

The solid figure analogous to a circle is the *sphere*. In Chapter 15 a circle was defined as follows:

> A circle is a simple closed curve having a point O in its interior and such that *if* A and B are *any* two points in the curve, $\overline{OA} \cong \overline{OB}$.

Since we used "simple closed curve" as an abbreviation for "simple plane closed curve" a circle always lies in a plane. If we simply change the requirement that all the points lie in a simple closed curve to say that they lie in a simple closed surface, we get a good definition of a sphere.

> A sphere is a simple closed surface having a point O in its interior and such that if A and B are *any* two points in the surface, $\overline{OA} \cong \overline{OB}$.

As in a circle, the point O is called the center of the sphere and the segments $\overline{OA}$ and $\overline{OB}$ are called radii. Figure 26-10 illustrates a sphere.

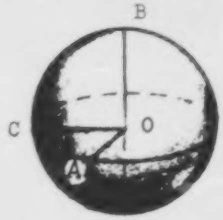


Figure 26-10. A sphere.

Technically the sphere is the surface of the solid and not the solid itself. However, we sometimes do refer to the solid itself for brevity as a sphere even though it should be called a spherical region. We shall be careful not to do this if any ambiguity might develop.

Many objects are spherical, that is, have the shape of a sphere. Some of these objects, such as ball bearings, are important to industry. Some, like rubber balls, are used as toys. It is because of these many spherical objects and, most of all, because of the shape of the earth which is almost a sphere, that it is important to know some of the properties of a sphere.

The surface of the earth is a fairly good representation of a sphere. But, it is not exactly a sphere because of its mountains and its valleys. Also, the earth is somewhat flattened at the poles. (The length of the equator is 24,902 miles and that of a great circle through the poles is 24,860 miles--like most mature bodies, it is slightly large around the middle!)

The surface of a basketball is also a good representation of a sphere. The surface of some Christmas tree ornaments, or the surface of a BB shot are even better representations of a sphere because they are smoother.

There are many interesting properties of a sphere of which we shall give a few. A sphere separates space so that any curve connecting a point in the interior to a point in the exterior must contain a point of the sphere.

A line may intersect a sphere in at most two points. If such a line passes through the center of the sphere, as $\overleftrightarrow{AB}$ does in Figure 26-11, the segment $\overline{AB}$ is called a diameter of the sphere.

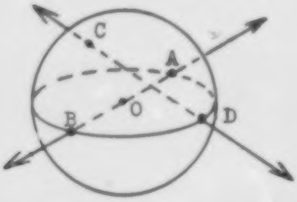


Figure 26-11.

In the same Figure $\overline{CD}$ is not a diameter since the center O is not in $\overleftrightarrow{CD}$.

Since the surface of the earth is approximately a sphere, maps of the earth are best drawn on globes where points may be identifed by lines of latitude and longitude. Maps drawn in planes also use lines of latitude and longitude, but the origin of these lines is much clearer if we consider a sphere to represent the surface of the earth.

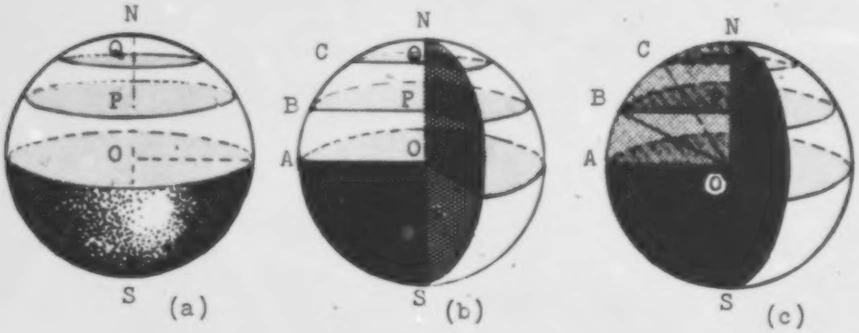
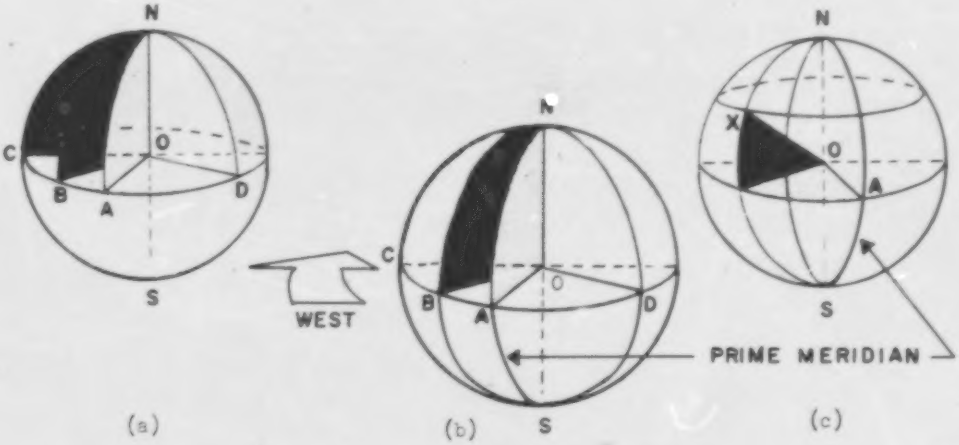


Figure 26-12. Lines of latitude on a sphere representing the earth.

In Figure 26-12 N and S mark the North and South poles of the earth. $\overline{NS}$ is called the earth's axis. Planes which are perpendicular to $\overline{NS}$ cut the surface in circles with centers at points in $\overline{NS}$ such as O, P and Q. See Figure 26-12. These circles are the circles of latitude. The one with center at O which is the center of the sphere, is called the equator. Figures 26-12b and 26-12c are cut-away pictures showing how we name the circles of latitude. In (c) if $m(\angle AOB) = 30$, we say the circle with center P is the circle whose latitude is 30° north of the equator. Any point on this circle is said to have 30° north latitude. Similarly if $m(\angle AOC) = 60$, any point on circle Q has 60° north latitude, and so forth for any other points. Each point lies on one such circle and has a fixed latitude north or south of the equator.

Lines of longitude are determined in a different fashion. This time we take a number of planes each of which contains the axis $\overline{NS}$. These cut



(a) (b) (c)

Figure 26-13. Lines of longitude on a sphere representing the earth.

the surface of the earth in circles such as those with the semi-circular arcs $\widehat{NAS}$, $\widehat{NBS}$ and $\widehat{NCS}$. These circles are the circles of longitude and the arcs are called meridians. In order to name them we pick one meridian which we will call the prime meridian. Let us pick $\widehat{NAS}$. In the cut-away drawing in (b) we show the various meridians cutting the equator at A, B, C, etc. If $m(\angle AOB) = 22$, we name the meridian $\widehat{NBS}$ as the meridian whose longitude is 22° West. Any point on $\widehat{NBS}$ has 22° West longitude.

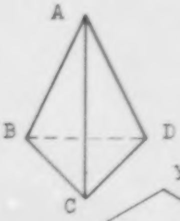
The point D is such that m($\angle AOD$) = 32. Any point on $\widehat{NDS}$ has 32° East longitude, since D is on the other side of A from B. In all maps the prime meridian is the one which goes through Greenwich, England. In (c) of Figure 26-13 the point λ has latitude approximately 40° North and longitude approximately 74° West. It therefore marks approximately the position of Philadelphia on the globe.

Wooden or plastic models of solids bounded by such figures as pyramids, prisms, cylinders, cones and spheres are very useful in studying these figures. When you feel the sharp point of a vertex, the line of an edge, the smoothness of a sphere, it is easier to grasp some of the ideas we have been discussing.

Exercises - Chapter 26

Name the solids pictured below.

1.



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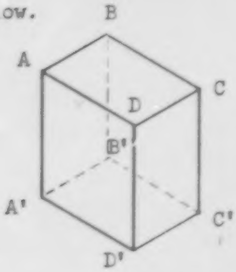
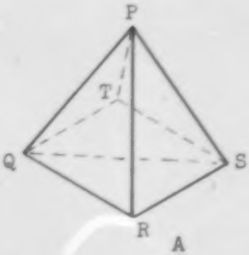
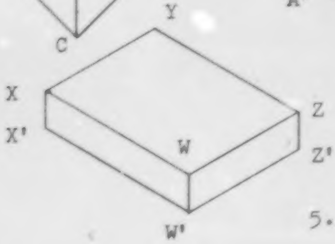


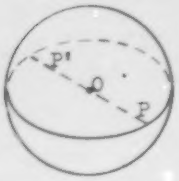
3.



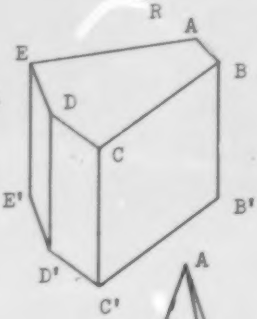
4.



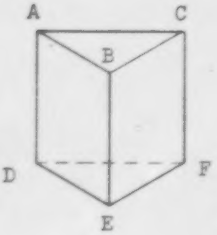
5.



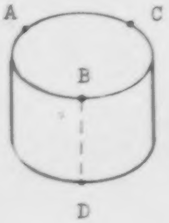
6.



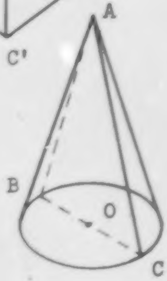
7.



8.

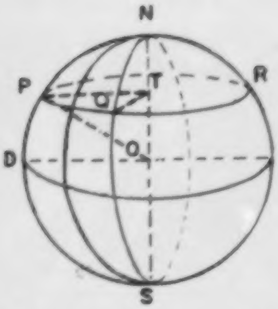


9.



10. In each figure identify one of each of the following as is appropriate. a) base, b) edge, c) vertex, d) center, e) radius, f) element, g) lateral face, h) diameter

11.



Identify the circle of latitude and circle of longitude for P on this globe. Indicate which angle measures the latitude of P. If NQS is the prime meridian indicate the angle which measures the longitude of P.

12. a) Sketch a prism with a square base.
    b) Sketch a triangular prism.
    c) Sketch a cylinder which is not a right cylinder.
    d) Sketch a pyramid with a base which is a quadrilateral.
    e) Sketch a cone.
    f) Sketch a sphere.

Solutions for Problems

1. They are all pyramids except for (e).
2. The only doubtful one is (c) in which the base is the region enclosed by the square.

Chapter 27

AREA

Introduction

In Chapter 16 we discussed the concepts of linear and angular measure, that is, how numbers can be assigned to segments and angles which measure them in terms of previously specified unit segments and angles. In this chapter and the next one, we want to extend the idea of measurement to plane and solid regions. What, then, do we mean by the measure of a plane region?

> To measure a plane region is to select a certain
> unit and to assign in terms of that unit a number
> which is called the *measure of the area* of the region.

Note that just as in the length of a line the area of a region involves both a number and a unit. Thus an *area* may be expressed as six square inches and written as 6 sq. in. The measure of the area is the number 6. While it is important to have these distinctions clearly in mind when working with lengths, areas and, later, volumes, it becomes too cumbersome to keep mentioning them. The important thing to remember is that we always compute with *numbers*, but we express answers in terms of numbers and the appropriate units.

Let us recall how the subject of linear measurement was approached since area will be approached in a similar manner. First we encountered the intuitive concept of *comparative length* for line segments: any two line segments can be compared to see whether the first of them is of *smaller length*, or the *same length*, or *greater length* than the second. Corresponding to this we have in the present chapter the idea of *comparative area* for plane regions. (Recall that by definition a plane region is the union of a simple closed curve and its interior.) Even when they are rather complicated in shape, two regions can, in principle at least, be compared to see whether the first of them is of *smaller area*, or the *same area*, or *greater area* than the second.

In the case of line segments, this comparison is conceptually very simple: we think of the two segments to be compared, say $\overline{AB}$ and $\overline{CD}$, as being placed one on top of the other in such a manner that A and C coincide: then either B is between C and D, or B coincides with D, or B is beyond D from C, etc. This conceptual comparison of line segments is also easy to carry out approximately using physical models (drawings and tracings, etc.) of the line segments involved.

Comparison of Regions

In the case of plane regions, this comparison is more complicated, both conceptually and in practice. This is because the shapes of the two plane regions to be compared may be such that neither will "fit into" the other. How, for example, do we compare in size (area) the two plane regions pictured below?

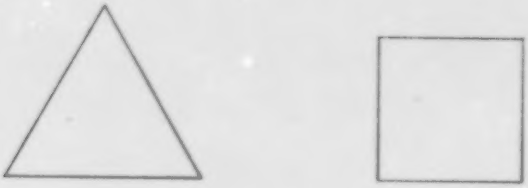
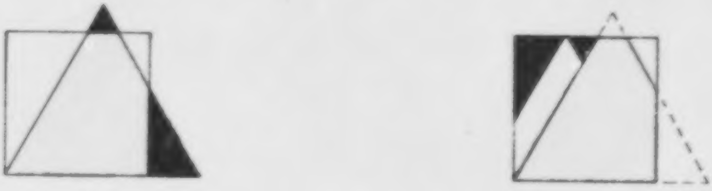


Figure 27-1.

If we think of these regions as placed one on top of the other, neither of them will fit into the other. In this particular case, however, we can think of the two pieces of the triangular regions which are shown shaded heavily in Figure 27-2a as snipped off and fitted into the square region in Figure 27-2b.



(a)

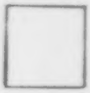
(b)

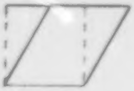
Figure 27-2.

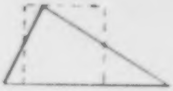
This shows that the triangular region is of smaller area than the square region. As the figures involved become more complicated in shape, this sort of comparison becomes increasingly difficult in practice. We need a better way of estimating the area of a region.

Problems *

1. Make models and compare these regions.







* Solutions for problems in this chapter are on page

2. Make models and try to compare these regions.

Units for Area

In studying linear measure we first found out how to compare two segments. What was the second step in the process? We chose a *unit* of length. That is, we selected a certain arbitrary line segment and agreed to consider its length to be measured, exactly, by the number 1. In terms of this unit we could then conceive of line segments of lengths exactly 2 units, 3 units, 4 units, etc., as being constructed by laying off this unit successively along a line 2 times, 3 times, 4 times, etc. The process of laying off the unit successively along an arbitrary given line segment yielded underestimates and overestimates for the length of the given segment since the segment might have turned out to be greater than 3 units (underestimate) but less than 4 units (overestimate). We selected the closer of these two estimates as the measure of the segment in terms of the selected unit, realizing that any such measure is usually only approximate and subject to error. Since the error was at most one-half of the unit used, by selecting a smaller unit, we found we could usually make the measurement more accurate.

We now proceed similarly in the measurement of area. The first step is to choose a unit of area, that is, a region whose area we shall agree is measured exactly by the number 1. Regions of many shapes, as well as many sizes, might be considered. An important thing about a line segment as a unit of length was that enough unit line segments placed end-to-end (so that they touch, but do not overlap) would together *cover* either exactly or with some excess any given line segment. Similarly, we need a unit plane region such that enough of them placed so that they touch, but do not overlap will together *cover* either exactly or with some excess any given plane region. Some shapes will not do this, for example, circular regions do not have this property. Thus, in Figure 27-3, if we try to cover a triangular region with small non-overlapping congruent circular regions, there are always parts of the triangular region left uncovered. On the other hand, we can always completely cover a triangular region, or any region, by using enough non-overlapping congruent square regions.

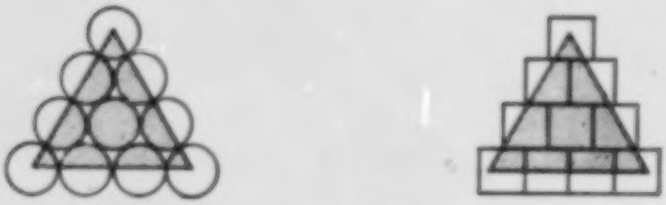


Figure 27-3.

While a square region is not the only kind of region with this covering property, it has the advantage of being a simply shaped region. The size of the unit of area is determined by choosing it as a square whose side has length equal to one linear unit. It then turns out that the use of such a square region as the unit of area makes it easy to compute the area of a rectangle by forming the product of the numbers measuring the lengths of its sides.

A Scale to Estimate an Area

Having chosen a unit of length, we then made scales and rulers to help in measuring the length of a given line segment. A corresponding instrument is not usually available for area, but we can easily make one for ourselves. This is a *grid* which is a regular arrangement of non-overlapping square unit regions as shown in Figure 27-4.

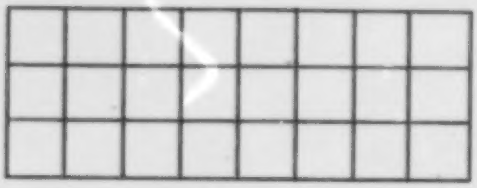


Figure 27-4. A grid.

To use such a grid for estimating the area of a given region, we think of it as superimposed on the region. This is illustrated in Figure 27-5. We can verify by counting that 12 of the unit regions pictured are contained entirely in the given region. These are the units heavily shaded in Figure 27-5a. This shows that *the area of this region is at least 12 units.* This is an underestimate. We can also verify by counting that there are 20 additional unit regions lightly shaded in Figure 27-5b, which together cover the rest of the region. Thus the entire region is covered by 12 + 20 or 32 units. This shows that the area of this region is at most 32 units. 32 is then an overestimate of the measure. That is,

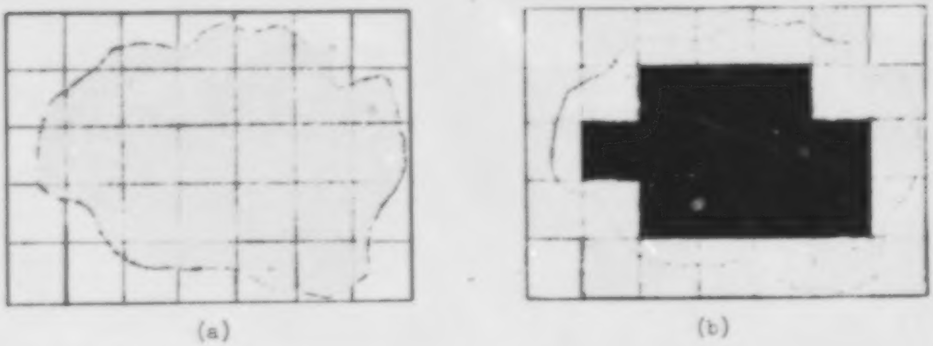


Figure 27-5. Using a grid to measure a region.

we now know that the area of the region is somewhere between 12 units and 32 units. Since the difference between the two estimates is 20 units, we see that the accuracy is not very good. The lightly shaded region in Figure 27-5b represents this difference.

In Chapter 16 we saw that more accurate estimates of lengths could be achieved by using a smaller unit. The same is true with area. To illustrate this fact, let us re-estimate the area of the same region in Figure 27-5, using this time the unit of area determined by a unit of length just half as long as before.

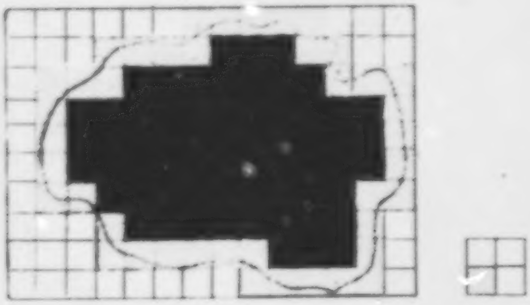


Figure 27-6. Using a new unit $\frac{1}{4}$ of the old unit.

We can verify by counting that there are 63 of the new unit regions pictured which are contained entirely in the given region. This shows that the area of the region is at least 63 (new) units. We can also verify by counting that there are 41 additional unit regions pictured which together cover the rest of the region. Thus, the entire region is covered by 63 + 41 or 104 of the new units. This shows that the area of this region is at most 104 (new) units. That is, we now know that the area of the region is somewhere between 63 (new) units and 104 (new) units.

Let us compare these new estimates of the area with the old ones. Each old unit contains exactly 4 of the new units, as is clear from Figure 27-6. Each new unit is $\frac{1}{4}$ of the old unit. Thus the new estimates are $\frac{1}{4} \times 63$ or $15\frac{3}{4}$ and $\frac{1}{4} \times 104$ or 26 old units as compared with our former estimates of 12 and 32. The difference is $10\frac{1}{4}$ old units compared to the former 20. Plainly the new estimates based on the smaller unit are the more accurate ones. This may still be quite unsatisfactory. However, in principle it would be possible to estimate the area of this region or even of regions of quite general shape to any desired degree of accuracy by using a grid of sufficiently small units in this way. In practice, the counting involved would quickly become very tedious. Furthermore, where drawings are used to represent the region and grid involved, we would, of course, also be limited by the accuracy of these drawings.

Basic Ideas of Area

Let us summarize the discussion to this point. Actually, the emphasis here is not so much on accurate estimates as it is to grasp the following basic sequence of ideas.

1. Area is in some sense a feature of a region (and not of its boundary).
2. Regions can be compared in area (smaller, same, greater), and regions of different shapes may have the same area.
3. Like a length, in theory, an area should be describable or measurable, exactly, by some appropriate number (not necessarily a whole number). Practically this is usually impossible. See Item 5 below.
4. For this purpose we need to have chosen a unit of area just as we earlier needed a unit of length.
5. The number which measures exactly the area of a region can be estimated approximately, from below and from above, by whole numbers of units.

6. In general, smaller units yield more accurate estimates of an area.

Formula for Area of a Rectangle

For some of the more common plane regions such as those whose boundaries are polygons or circles, formulas can be found to compute the measure of the area in terms of the linear measure of appropriate line segments of the figure. The careful derivation of these formulas is a problem for a more advanced course. But the drawings of a few figures will show the plausibility of some of them.

For the rest of this chapter, whenever we want to refer to the area of the plane region bounded by a certain figure we shall, for brevity, refer to it as the area of the figure. Thus instead of saying the "area of the triangular region bounded by $\triangle ABC$" we shall say the "area of $\triangle ABC$," and so on. Admittedly this is slightly inaccurate and if there is any possibility of ambiguity we will go into detail.

If the sides of a rectangle are measured in terms of the same unit, we may find that the lengths of the sides are *a* and *b* units where *a* and *b* are whole numbers. We then have an *a* by *b* array of unit squares and we know by the definition of multiplication of whole numbers that the number of squares in the array is $a \times b$. We can check this by an actual count if we care to. In any case, the rectangle contains $a \times b$ units of area and we say the measure of the rectangle is $a \times b$. Thus for this case the measure of area equals the product of the measures of two sides, and we write $A = a \times b$. See Figure 27-7.

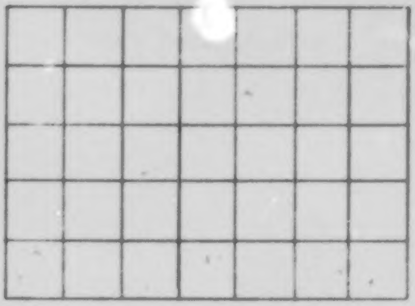
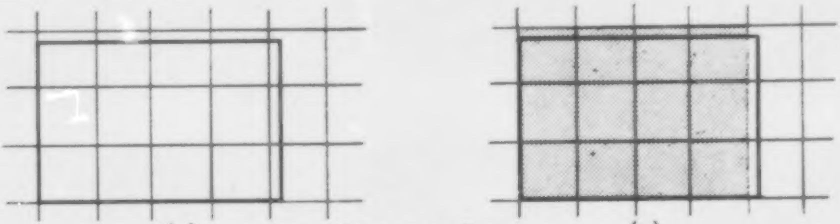


Figure 27-7. $A = 5 \times 7 = 35$.

Practically, of course, we can measure the sides of a rectangle only to the nearest unit. In Figure 27-8, the length and width of the heavily outlined rectangle may measure 4 and 3 inches respectively to the nearest unit.



(a) (b)

Figure 27-8.

By counting unit regions in the superimposed grid we see that the area is between 8 and 15 square inches. If we use the measures of the sides to the nearest inch in the formula

$$A = a \times b,$$

we would get $A = 3 \times 4 = 12$ and the area would be 12 square inches. This would be the exact area of the region shaded in Figure 27-8b which does seem to have about the same size as the given rectangle and it lies between the underestimate 8 and the overestimate 15 that we got using the grid. It is probably not, of course, the exact area of the original rectangle but it is the measure of this area to the nearest square inch. If we use a smaller unit, say one $\frac{1}{10}$ of an inch and get the measure of length and width to be 4.2 and 2.8 then in terms of this unit, their product, 11.76, would again lie between the underestimate and overestimate we would get using a $\frac{1}{10}$ inch grid. Since the unit of length is now .1 inch, the unit of area is .01 square inch and the number 11.76 is the measure of area to the nearest hundredth of a square inch. We may never get an exact measure of the area, but by using smaller units we can usually make our error smaller.

On the basis of these experimental results we make the definition:

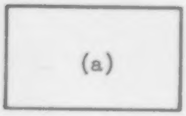
> The *measure of the area* of a rectangle is the number obtained as the product of the number measuring the base and the height. The smaller the unit of length used, the more accurate the measure of area will be.

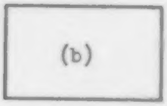
We usually let A stand for the measure of the area and b and h for the measures of the base and height. For brevity we usually say "the area is the product of the base and height" even though we should say "measure of" each time, and we write the formula for the area of any rectangle as:

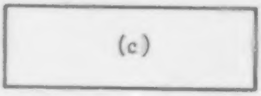
$$A = b \times h.$$

Problems

3. Make a tracing of the grid in Figure 27-4 and by superimposing it on each of the following rectangles make under- and overestimates of their area.



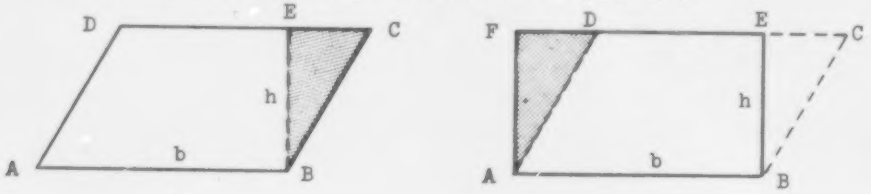




4. Measure the base and height to the nearest unit and determine the area by using the formula $A = b \times h$.
5. Make a new grid by dividing each unit of length in half as in Figure 27-6 and use this to estimate the areas of the figures in Problem 3.
6. Do the same as in Problem 4 using the new unit. Convert your results to the units of Problem 4 and compare the results.

Area of a Parallelogram and a Triangle

We now consider a parallelogram ABCD. See Figure 27-9.



(a) (b)

Figure 27-9.

By cutting off $\triangle BEC$ from parallelogram ABCD and moving it over next to $\overline{AD}$, we can see that the measure of area of parallelogram ABCD is equal to the measure of area of rectangle ABEF.

Therefore our formula $A = b \times h$ holds for any parallelogram if by h we understand the measure of the height $\overline{BE}$ and not the side $\overline{BC}$ of ABCD, i.e., the vertical distance and not the "slant" distance.

The formula for the area of a triangle ABC follows from that of a parallelogram. If a model of $\triangle ABC$ is made and labelled A'B'C', it can be turned over and placed alongside $\triangle ABC$ to form the parallelogram ABA'C whose base and height are $\overline{AB}$ and $\overline{CD}$. $\triangle ABC$ is thus one-half of parallelogram ABA'C and its area must be half that of the parellelogram.

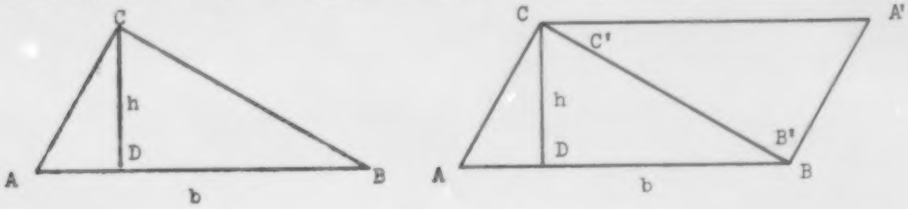


Figure 27-10.

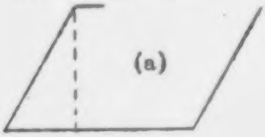
In the same fashion any triangle is half of a parellelogram whose base and height are identical with that of the triangle. Therefore, for a triangle

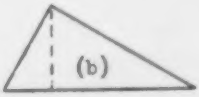
$$A = \frac{1}{2} \times b \times h.$$

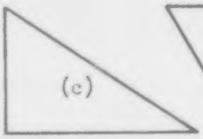
Any other polygon can be divided into triangles by suitably drawn segments and, therefore, its area may be found.

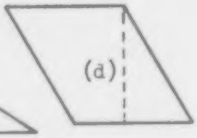
Problems

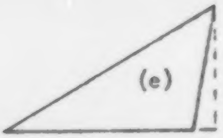
7. Use a centimeter scale to determine the base and height of the following figures and find their areas.

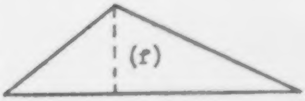


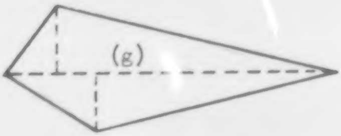












8. Use an eighth-inch scale to find the areas of the same figures. Which results are more accurate, those in Problem 7 or 8?

Area of a Circle

The area of a circle may be estimated on a grid just as we estimated the area of an irregular simple closed region. A formula for this area however is more difficult to obtain than those for polygonal regions. All our formulas have been derived for polygonal regions which have line segments as their boundaries. The circle has no segments on its boundary. We can make some progress, however, if we cut up the circular region into a number of congruent parts and rearrange them. Thus if we cut it into 4 parts, we get Figure 27-11.

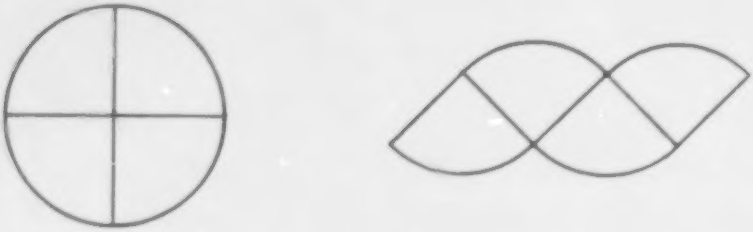


Figure 27-11. Circle cut into 4 parts and rearranged.

If we cut it into 8 and 16 parts we get Figures 27-12 and 27-13.

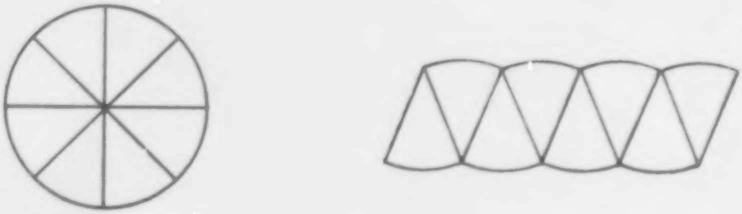


Figure 27-12. Circle cut into 8 parts and rearranged.

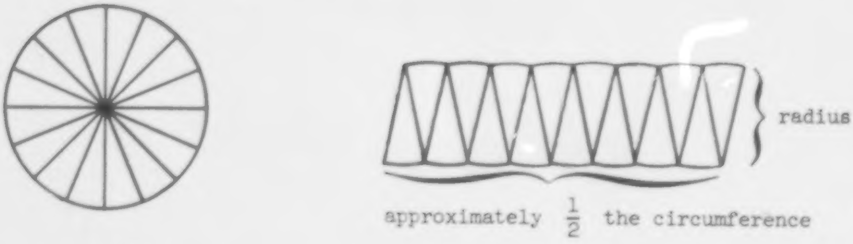


Figure 27-13. Circle cut into 16 parts and rearranged.

We see that as we divide the circle into more and more parts, the resulting figure looks more and more like a parallelogram. The base of this figure is nearly equal in length to half the circumference as it is made up of the arcs of half the pieces. The height of the figure is more and more nearly equal to the length of the radius. Thus the area of the figure can be approximated as the product of one-half the circumference by the radius. Since the area of this figure is the same as that of the circle, we get the formula

$$A = \frac{1}{2} \times C \times r.$$

This formula can be checked by using models of circles such as the bases of tin cans. Drawing a circle on a grid such as that of a piece of graph paper with units $\frac{1}{10}$ inch in length, the area can be estimated fairly accurately. If the circumference and radius are represented by a piece of string and these in turn measured on the graph paper scale, the results obtained by formula and by estimate can be compared.

The diameter of any circle is twice as long as its radius. Suppose we wish to compare the circumference of a circle to its diameter. This comparison is best made by considering their ratio. Measuring the circumference and diameter of a model we may try to express their ratio as the ratio of two whole numbers. This ratio turns out to be a little larger than 3:1. If the lengths are measured with more and more accuracy it will be found that the ratio is successively 3.1:1, 3.14:1, 3.142:1, 3.1416:1, etc. As a matter of fact this ratio can never be expressed as the ratio of two whole numbers and is therefore not a rational number. Numbers which are not rational will be studied briefly in Chapter 30, but meanwhile, assuming that the successive approximations made above are approximations to some number, we call this number π (read pi) and say $C:d = \pi:1$. This proportion holds for any circle. It may be written in fraction form as

$$\frac{C}{d} = \frac{\pi}{1}$$

and yields the formula $C = \pi \times d$ or the better known formula

$$C = 2 \times \pi \times r.$$

Combining this with the formula for the area of a circle we may express the latter as

$$A = \frac{1}{2} \times C \times r = \frac{1}{2} \times 2 \times \pi \times r \times r,$$

or

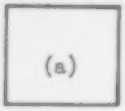
$$A = \pi \times r \times r.$$

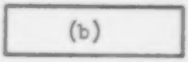
For practical purposes, when measurement of segments are made even to the nearest hundredth of an inch, a sufficiently good approximation to π is $\frac{22}{7}$ since this is equal to 3.143 ... which to the nearest hundredth agrees with π.

Exercises - Chapter 27

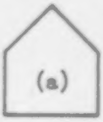
Tell which simple closed curve of each pair has the region with the greater area. (You may make a paper model of one of these regions to see if the pieces can be placed, without overlapping, on the other region.)

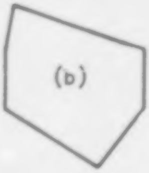
1.



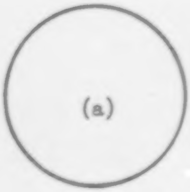


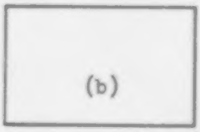
2.





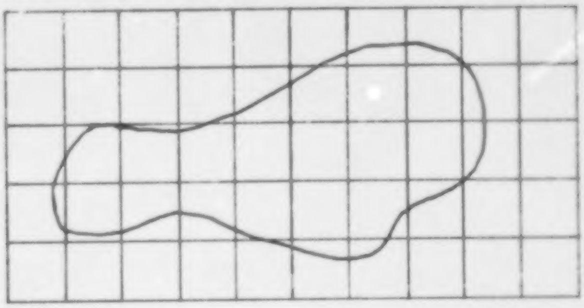
3.





4. Which plane region has the greater area - a region bounded by a square with a side whose length is 3 inches or a region bounded by an equilateral triangle with a side whose length is 4 inches? You will need models of these regions.

5. Consider the region pictured below on a grid of units.



Fill in the blanks:

(a) There are _____ units contained entirely in the region.

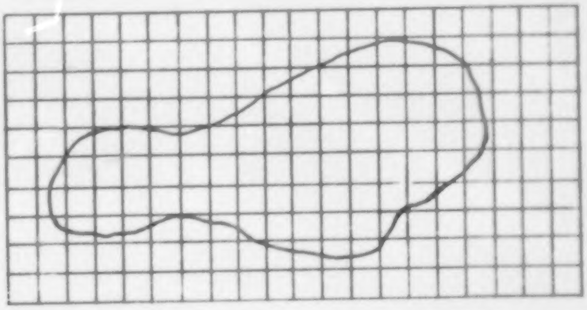
(b) There are _____ units needed to cover the region completely.

(c) The area of the region is at least ______ units and at most ______ units.

(d) The difference is _____ units.

6. Let us choose a new unit of area. A square region has as its side a segment just half as long as before. For every old unit of area we will then have 4 new units of area.

Consider the same region pictured below on a grid of new units.



Fill in the blanks:

(a) There are ____ units contained entirely in the region.

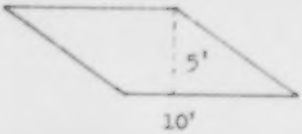
(b) There are ____ units needed to cover the region completely.

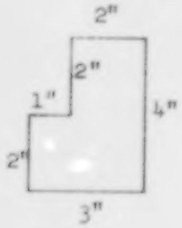
(c) The area of the region is at least ____ units and at most ____ units.

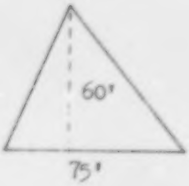
(d) The difference is ____ units.

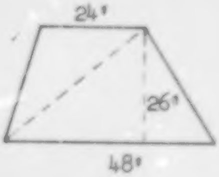
(e) Since each new unit is $\frac{1}{4}$ the old unit, this difference in terms of the old units is ____ units.

7. The following figures are scale drawings of larger ones. The dimensions of each are given. Find the areas of the original figures.









8. A room measures 15'3" by 20'9". What is its area in square inches? What is its area in square feet?

9. If the room mentioned in Problem 8 is measured to the nearest foot, what is its length and width? What is its area in square feet? Why is the answer different from that in Problem 8?

10. Take an ordinary silver 50¢ piece. Trace an outline of it on a graph paper grid with unit $\frac{1}{10}$ inch. Estimate the area by using the grid. Use thread to represent the circumference and radius and measure them on the graph scale. Use the formula to obtain the area. Compare the two results.

11. The diameter of a circle is measured as 4.2 inches. Find the circumference and area.

12. The radius of a circle is 12 feet. Find the area.

Solutions for Problems

1. Models of the second and third figures may be cut on the dotted lines and made to match a model of the first figure.

2. A model of the circle will fit inside the square; thus the area of the circle is less than that of the square.

3.

	Underestimate	Overestimate
a.	3 sq. units	6
b.	2	6
c.	4	10

4. a. $2 \times 3 = 6$
   b. $2 \times 3 = 6$
   c. $1 \times 4 = 4$

5.

	Underestimate	Overestimate
a.	18 new units or $4\frac{1}{2}$ old units	24 new units or 6 old units
b.	15 new units or $3\frac{3}{4}$ old units	24 new units or 6 old units
c.	16 new units or 4 old units	27 new units or $6\frac{3}{4}$ old units

6. a. $4 \times 6 = 24$ new units or 6 old units compared to 6 before.
b. $3 \times 5 = 15$ new units or $3\frac{3}{4}$ old units compared to 6 before.
c. $3 \times 9 = 27$ new units or $6\frac{3}{4}$ old units compared to 4 before.

7. a. $A = 3 \times 2 = 6$
b. $A = \frac{1}{2} \times 3 \times 1 = 1\frac{1}{2}$
c. $A = \frac{1}{2} \times 3 \times 2 = 3$
d. $A = 2 \times 2 = 4$
e. $A = \frac{1}{2} \times 3 \times 2 = 3$
f. $A = \frac{1}{2} \times 4 \times 1 = 2$
g. $A = (\frac{1}{2} \times 5 \times 1) + \frac{1}{2}(5 \times 1) =$
$= \frac{5}{2} + \frac{5}{2} = 5$

In each case the answer is: a. The area is 6 sq. cm.
b. The area is $1\frac{1}{2}$ sq. cm., etc.

8. a. $A = \frac{9}{8} \times \frac{6}{8} = \frac{54}{64}$
b. $A = \frac{1}{2} \times \frac{9}{8} \times \frac{4}{8} = \frac{36}{128} = \frac{18}{64}$
c. $A = \frac{1}{2} \times \frac{8}{8} \times \frac{6}{8} = \frac{48}{128} = \frac{24}{64}$
d. $A = \frac{5}{8} \times \frac{5}{8} = \frac{25}{64}$
e. $A = \frac{1}{2} \times \frac{6}{8} \times \frac{9}{8} = \frac{54}{128} = \frac{27}{64}$
f. $A = \frac{1}{2} \times \frac{13}{8} \times \frac{4}{8} = \frac{26}{64}$
g. $A = (\frac{1}{2} \times \frac{16}{8} \times \frac{3}{8}) + (\frac{1}{2} \times \frac{16}{8} \times \frac{3}{8}) = \frac{24}{64} + \frac{24}{64} = \frac{48}{64}$

The results in Problem 8 are more accurate since the unit $\frac{1}{8}$ inch is smaller than the unit 1 cm. used in Problem 7.

Note: Because of the necessary approximations in drawing and tracing scales, your answers to the problems in this chapter may differ from those given.

Chapter 28

MEASUREMENT OF SOLIDS

Introduction

The discussion of volumes of solid regions is more difficult than that of areas of plane regions primarily because it is hard for us to visualize solid regions when our pictures and diagrams are always in a plane. If you could make cardboard models of some of the solid regions, it might help you to see what is going on. Let us review again some of the concepts of measure of length and area.

In Chapter 16, linear measure was discussed. Linear measure is the assignment of numbers to line segments by which to "measure" their "length." These numbers are assigned to compare the segment with a previously selected unit of length. Thus, if the unit is an inch, a particular segment $\overline{AB}$ may be more than 5 inches and less than 6 inches in length. If it is nearer to 5 inches than to 6, the measure assigned is 5. By using smaller and smaller units the measure can be made more and more accurate. Using rational numbers or decimals, the measure of a segment can be expressed as accurately as desired. Thus by using a unit $.1$ inch long, the length of $\overline{AB}$ might be 5.1 inches, using a unit $.01$ inch long the length might be 5.14, using a unit $.001$ inch long the length might be 5.138 and the process could be continued using still smaller units if greater precision were wanted.

Theoretically, a segment may have an exact length, practically it never has. We are talking theoretically when we say a segment has a length. We are talking practically when we say its length is a particular measure accurate to a certain number of places.

The length of a curve has not been discussed before except for curves made up of one or more segments. However, the process for determining the length of a simple curve is to approximate the curve as closely as we please by a broken straight line and measure the length of the line as we did the

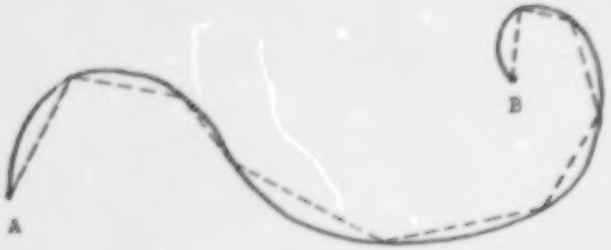


Figure 28-1. A curve approximated by a number of segments.

perimeter of a polygon. See Figure 28-1. Curves for which this cannot be done do exist, but they are so complicated to define and hard to work with that we will leave their discussion to the professional mathematicians.

When we discussed areas of plane regions in Chapter 27, we ran into difficulties similar to those with length. We found that a unit of area was a square 1 unit on a side. Putting many such squares side by side formed a grid and we could put our region on the grid and count how many squares were totally inside, say n, and how many were necessary to totally cover it, say N. Then the area was a number somewhere between n and N which we could approximate better and better by using smaller and smaller unit squares.

The problem of actually computing the area we solved only in a few cases. We found that the area of a rectangle is measured by a number which is the product of the measures of the base and the height. This is true even if the unit squares don't fit exactly. The area of a parallelogram can be computed by making an equivalent rectangle. The area of a triangle is half that of a certain parallelogram and the area of any polygon can be computed by dividing it into triangles. Formulas for the areas of certain other figures can be derived easily, but for most plane regions the simplest method for determining their area is still to approximate them on a rectangular grid, preferably with fairly small divisions.

Volume of a Solid Region

When we talk about solids and want to measure them, we have to think about the amount of space they occupy. One way to approach this is again through the idea of a rectangular grid, but this time in space rather than in a plane. See Figure 28-3. The rectangular grid is made up of unit solids which are cubes, one unit in each dimension. The unit of length is a line segment one unit long; the unit of area, a square one unit in each of two dimensions; and the unit of volume is a cube one unit in each of three dimensions. See Figure 28-2. The grid in Figure 28-3 is an outline of the solid cubes we imagine.

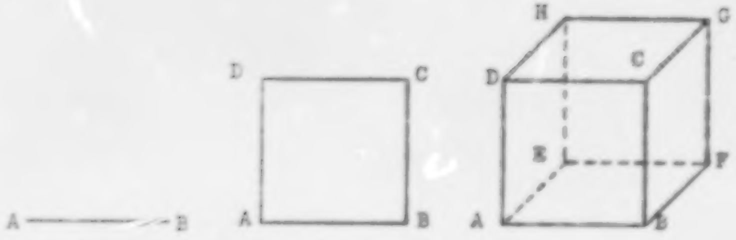


Figure 28-2. Units of length, area, and volume.

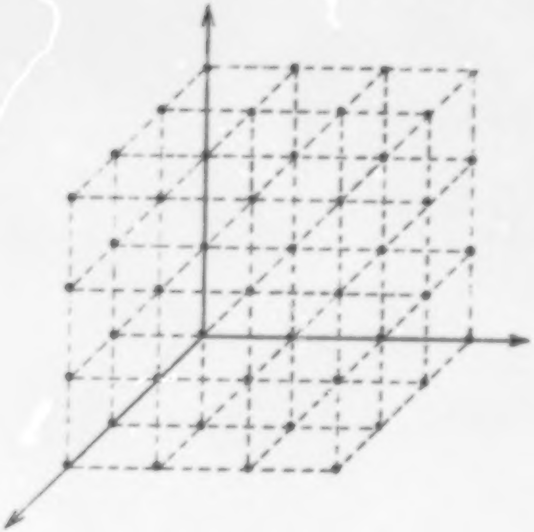


Figure 28-3. An outline of a rectangular grid for space.

If any solid is given, a grid such as in Figure 28-3 may be imagined to surround it. A certain number n of the solid units may be completely enclosed by the solid and a larger number, N, of them will completely enclose it, thus giving an underestimate and an overestimate for V which represents the measure of the volume in terms of the chosen unit.

$$n < V < N$$

By using smaller and smaller units we can make better and better approximations to V much as was the case with measurement of length and area.

Let us now turn from this consideration of measuring the volume of any solid no matter how curiously shaped to consideration of the more regular and common solids. For these we will work out formulas for computing their volumes in terms of the linear measures assigned to segments associated with them, rather than approximating their measure by using space grids and counting procedures.

Volume of a Rectangular Prism

Remember that for brevity the words "volume of a rectangular prism" are used instead of the more accurate "volume of the solid region enclosed by a rectangular prism" and similarly for other solid figures.

Consider a large number of unit cubes. If we arrange three such cubes in a row and make two rows of them, we have a layer of 6 cubes. If 4 such layers are piled on top of each other, the result is a rectangular prism made up of 24 unit cubes. Its volume is 24 cubic units. See Figure 28-4a. Thus for a rectangular prism constructed in this way the measure of the volume seems to be $(2 \times 3) \times 4$. In general, for such a constructed prism, $V = (\ell \times w) \times h$, where ℓ, w and h stand for the whole unit measures of the length, width and height.

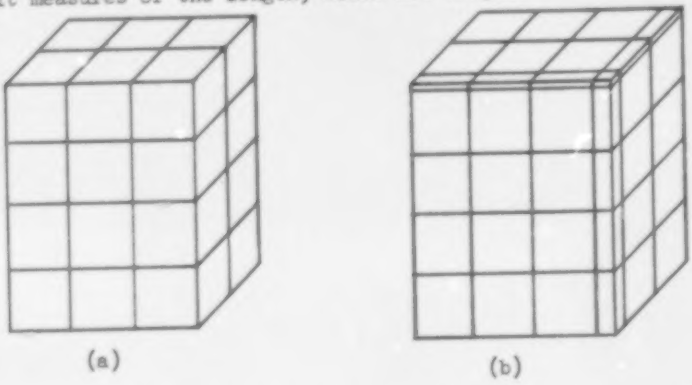
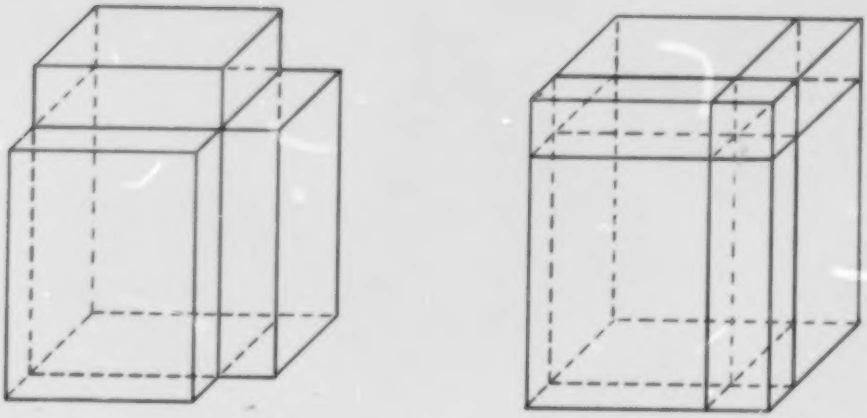

(a) (b)

Figure 28-4. Two rectangular prisms.

On the other hand, if some rectangular prism is given to us rather than our constructing it from cubes, the lengths of the three edges will not usually be an exact number of units. See Figure 28-4b. Nevertheless, any rectangular prism can be thought of as containing a certain number n of unit cubes in it. Perhaps there is some space left over. In this case, the volume is more than n which is therefore an underestimate of the volume. If we add a layer of cubes on each of the three adjacent faces and also add the cubes necessary to fill out to a rectangular solid with N cubes, as in Figures 28-5a and 28-5b, we will find that the new one contains the given solid and therefore N is an overestimate of the volume. We say the original solid has volume V and $n < V < N$. Just as in the case of area, we can make closer approximations to V by using a unit cube with edges say $\frac{1}{10}$ or $\frac{1}{100}$ those of the original unit. We can make the approximation to V as accurate as we please. What will happen is this: as the length of the unit segment gets smaller and smaller, the approximations to the length, width and height of the solid get better and better, the unit cubes



(a)

(b)

Figure 28-5. Filling out a rectangular prism.

get smaller and smaller, the difference between the underestimates and overestimates of V in terms of the original unit gets smaller and smaller and so the approximation to V gets better and better. We are then led to the definition:

> The volume of a rectangular solid is measured by the number $\ell \times w \times h$, where ℓ, w and h represent the measures of the length, width and height in the same units.

$$V = \ell \times w \times h.$$

Since the area of the base is equal to $\ell \times w$ we frequently say:

> The volume of a rectangular prism is the product of the area of its base by its height.

As a formula, letting B stand for the area of the base $\ell \times w$, this becomes:

For any rectangular prism: $V = B \times h$.

It should be noted that V stands for a number, the measure of the volume. The volume itself should always be expressed as a number together with the correct cubic units. Thus if in Figure 28-4 the units are inches, the volume should be given as 24 cubic inches.

Problems*

1. A child measures a rectangular prism with a ruler whose unit is an inch. The length is 5 inches, the width 3 inches and the height 6 inches. What is the volume?
2. The same prism as in Problem 1 is measured with a ruler whose unit is .1 inch. The length is now reported as 5.2, the width as 3.4 and the height as 6.3 inches. What is the volume? Note that the answer in Problem 1 was less than V. Is the answer to Problem 2 less than or more than the exact value of V or can we tell?

Properties of Volume

The volumes of many other solids are determined by comparing them with volumes of appropriately chosen rectangular solids. We are not going to make any attempt to get formulas for the volumes of many different solids. What we do want to emphasize is that volume is a number associated with

*Solutions for problems in this chapter are on page 387.

a solid region such that:

1. Volume is associated with the solid region and not with the closed surface which bounds the region.
2. Solid regions can be compared in volume and regions of different shapes may have the same volume.
3. Just as with length and area, in theory a volume is measurable exactly by a number. Practically this is usually impossible. See Item 5 below.
4. For this purpose, we need to choose a unit of volume just as we earlier needed a unit of length or area.
5. The number of units which measures exactly the volume of a region can be estimated approximately from below and above by whole numbers of units.
6. In general, smaller units yield more accurate estimates of a volume.

Formulas for Volumes of Certain Solids

Although the concept of volume resulting from the properties just mentioned is probably the most valuable idea to get, nevertheless we sometimes actually want to compute the number which measures the volume of a certain solid. Perhaps it will be well to determine the formulas for the volumes of a few of the more familiar solids. We already have the formula for the measure of the volume of a rectangular prism. $V = \ell \times w \times h$. This solid is a special case of a right prism (see Chapter 26). If you cut such a rectangular prism in half, see Figure 28-6, by the plane WYQS you get a triangular right prism whose volume is half that of the rectangular solid.

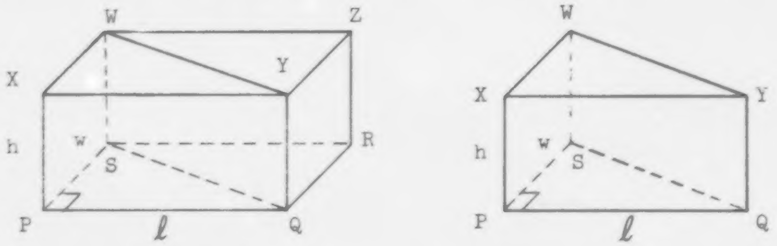


Figure 28-6. Rectangular and triangular prisms.

The measure of its volume is therefore given by

$$V = \frac{1}{2} \times (\ell \times w \times h).$$

But this may be rewritten as $V = (\frac{1}{2} \times \ell \times w) \times h$. The part of this formula which is in parentheses is the formula which gives us the area of the triangle PQS which is the base of the triangular prism. See Figure 28-7 where by looking straight down on the prism it may be seen that $\overline{PQ}$ is the base of $\triangle PQS$ and $\overline{PS}$ is its altitude so that the formula for the area of a triangle from Chapter 27 applies. Thus for this triangular right prism,

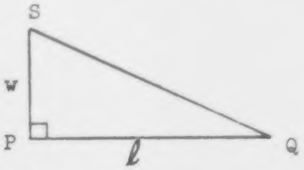


Figure 28-7. $\frac{1}{2} \times w \times \ell$ = area of $\triangle PQS$
= area of base of the triangular prism.

if B is used to represent the area of its base, the formula for the volume is $V = B \times h$ just as for the rectangular prism. This particular prism happens to have a right triangle for its base. Other right prisms may have scalene triangles for their bases or quadrilaterals or in fact any polygons. Physical models of several prisms can be made out of thin plywood or cardboard with open tops so that the solid region they enclose may be filled with sand. Suppose such prisms have the same height, h, and the polygons which are their bases have the same area, B, even though they may not have the same shape. Filling one model with sand and then pouring the sand from it into the other models demonstrates very convincingly that they all have the same volume. This may be written:

For any right prism: the volume is equal to
the area of its base times its height.

$$V = B \times h.$$

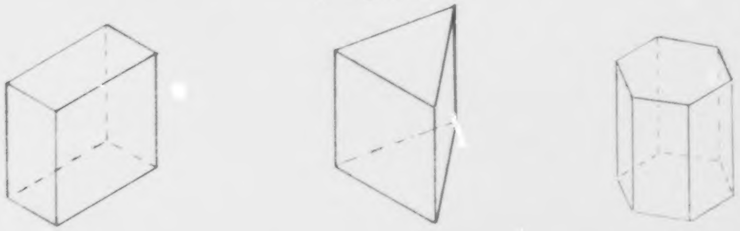
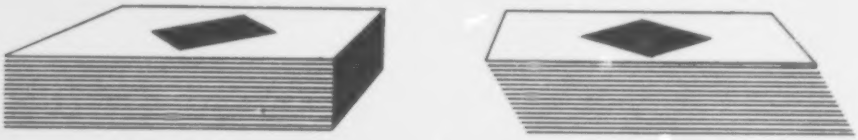


Figure 28-8. Different shaped prisms with the same volume.

If the prism is not a right prism, a good physical model of the situation is a deck of cards which has been pushed into an oblique position as in Figure 28-9b. Putting the deck back into a straight stack does not seem to change the amount of space occupied by them. It looks as though the formula $V = B \times h$ is a valid formula to find the volume of any prism and in fact it is. What we must be careful to note, however, is that the *h* in this formula represents the actual height of the prism and not a lateral edge. In Figure 28-10 the base of the prism is the parallelogram ABCD and the height is the measure of $\overline{PQ}$, a segment from a point P in the upper base perpendicular to the plane in which ABCD lies. $\overline{PQ} < \overline{PA}$ where $\overline{PA}$ is a lateral edge. We can now say:



(a) (b)

Figure 28-9. Model showing a right prism (a) with same value as oblique prism (b).

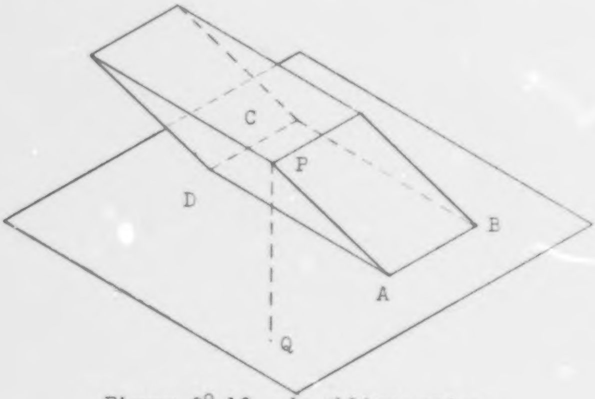


Figure 28-10. An oblique prism.

For *any* prism: $V = B \times h$.

In Chapter 26 it was stated that a prism is a special case of a cylinder. In fact, any cylinder can be approximated by prisms whose heights are the same as that of the cylinder and whose bases are polygons which approximate the bases of the cylinder. We guess that the formula $V = B \times h$ still holds and indeed it does.

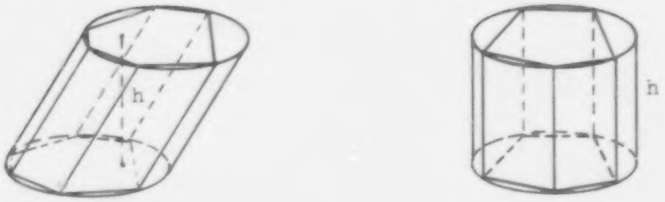
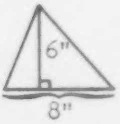


Figure 28-11. Cylinders approximated by prisms.

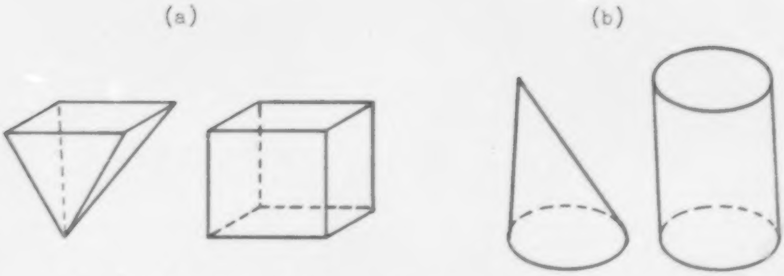
Of course, B stands for the area of the base of the cylinder and $\underline{h}$ for its height. Thus

For any cylinder: $V = B \times h$.

Problems

3. A triangular prism has as its base the triangle . The height of the prism is 12". What is the volume?
4. A tin can has height 3 inches and a base of area 12 square inches. What is its volume?
5. A truck is called a 5 ton truck if its capacity is 5 cubic yards. How big is a dump truck which has a body 6 feet wide by 9 feet long by 5 feet high?
6. A circular drum has a height of 35 inches and a base whose radius is 12 inches. What is its volume? (Use the area of a circle formula from Chapter 27.)

Volumes of solid regions bounded by pyramids and cones are hard to find formulas for in the way we have been proceeding. But, an experiment gives the formulas quite easily. If you take a certain pyramid and make a model of it and of a prism with base and height congruent to those of the pyramid, you will find that if you fill the pyramid with sand and pour it into the prism, the prism will be filled after three such pourings. The same is true for a cone and the corresponding cylinder.



(a) V of a pyramid is one-third V of a corresponding prism.

(b) V of a cone is one-third V of a corresponding cylinder.

Figure 28-12.

This leads to the formula:

For any pyramid or cone: $V = \frac{1}{3} \times B \times h$.

Problems

7. A pyramid has square bases with sides each 12 feet and height of 10 feet. What is its volume?
8. Look up the dimensions of the Great Pyramid of Cheops in Egypt. What is its volume?
9. A cone has height 12 feet and base a circle of area 6 square feet. What is the height of a cylinder whose base and volume are equal to that of the cone?
10. A cone has height 10 inches and base a circle of radius 3 inches. What is its volume?

The Volume of a Sphere

The last common solid to consider is the sphere. If the radius of a sphere is r, think of a right circular cone and a right circular cylinder each with the same radius r and each with height equal to the diameter of the sphere which of course is $2 \times r$. Consider hollow models of each.

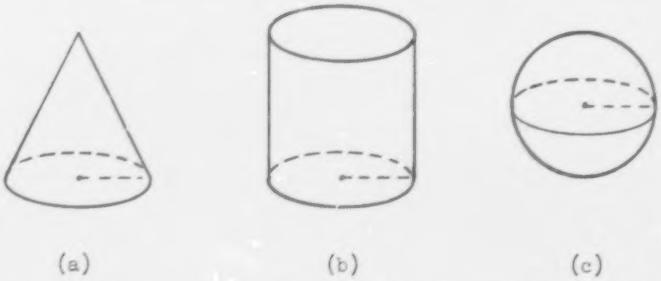


Figure 28-13. A cone, cylinder and sphere of radius r and height $2 \times r$.

If the cone is filled with sand and this sand is poured out of it and into the cylinder, the cylinder will be one-third full. If now the sphere is also filled with sand and then emptied into the cylinder which is already one third full with the sand from the cone, the cylinder will be completely full. Thus the volume of the sphere is just two-thirds that of the corresponding cylinder and just twice that of the corresponding cone. Since the radius of the base of the cylinder is r and its height is $2 \times r$ the volume, which is $B \times h$, is

$$V = (\pi \times r \times r) \times (2 \times r).$$

Therefore, the volume of the sphere is

$$V = \frac{2}{3} \times (\pi \times r \times r) \times (2 \times r)$$

or

$$V = \frac{4}{3} \times \pi \times r \times r \times r.$$

Problems

11. Find the volume of a sphere whose radius is 3 inches.
12. Find the volume of a raindrop whose radius is .1 inch.

Surface Areas of Solids

Not only is the volume of a solid of interest but frequently so also is the area of its surface. This area may be found quite easily for certain solids, such as right prisms, whose surfaces have the following property. If a cardboard model is made of the surface of the right prism in Figure 28-14, it may be cut along the lateral edge $\overline{AA'}$ and then along the edges of the upper and lower bases $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EA}$, $\overline{B'C'}$, $\overline{C'D'}$, $\overline{D'E'}$ and $\overline{E'A'}$. The

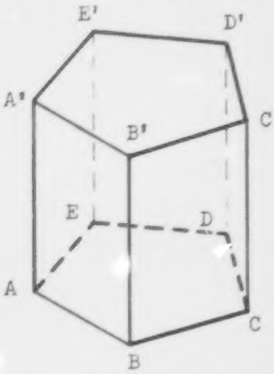


Figure 28-14. Model of a right prism cut to be flattened out.

model of the surface can now be flattened out on a plane in the form shown in Figure 28-15.

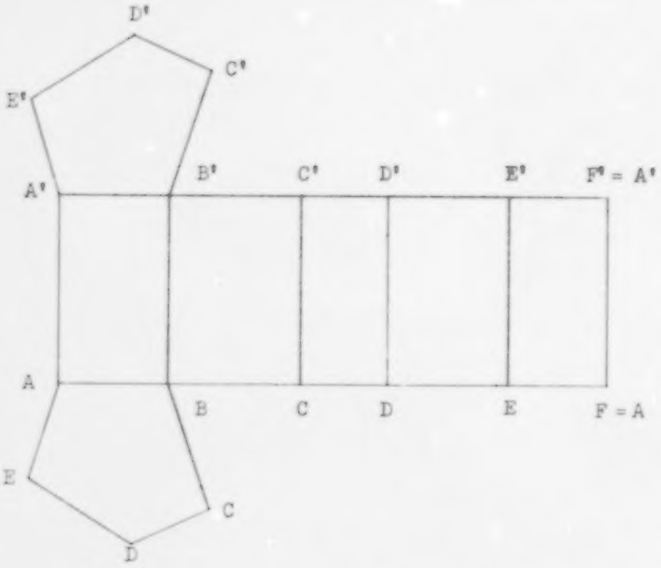


Figure 28-15. Surface of a right prism flattened out in the plane.

It can be seen that the surface is made up of two congruent polygons which are the bases as well as a number of rectangles each a lateral face which together make up the lateral surface. Each of the rectangles has the same height, h, as the prism and their union is the rectangle whose base $\overline{AF}$ is the union of all their bases. The area of this rectangle, which is equal to the area of the lateral surface of the prism, is called the lateral area. It is equal to its height, h, times the measure of $\overline{AF}$. But the measure of $\overline{AF}$ is by definition the perimeter of the base of the prism. Thus the total surface area of a right prism is equal to the sum of the areas of its two bases plus its lateral area. As a formula this can be written

$$\text{Surface area} = (2 \times B) + (h \times P)$$

where B stands for the area of the base, h the height of the prism and P the perimeter of the base.

The surface area of a right circular cylinder can be developed similarly and the result is almost the same. We cut out the bases, cut along an element of a model and flatten it out as in Figure 28-16.

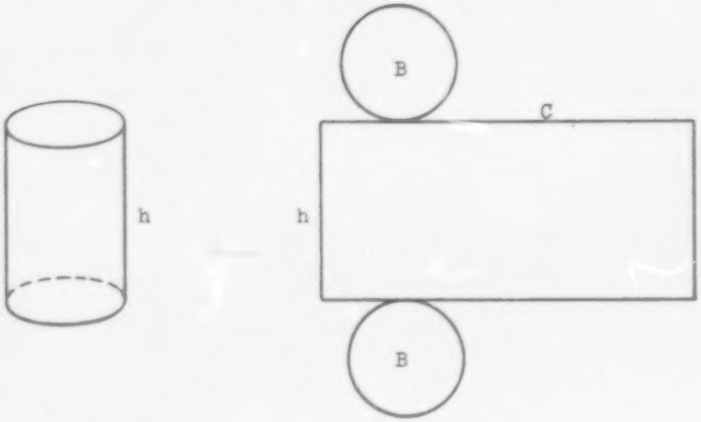


Figure 28-16. The surface area of a right circular cylinder.

Hence, we find

$$\text{Surface area of a cylinder} = (2 \times B) + (C \times h)$$

where B and h are as before and C stands for the circumference of the base. Expressing B and C in terms of π and r we get

$$\begin{aligned}\text{Surface area} &= (2 \times \pi \times r \times r) + (2 \times \pi \times r \times h) \\ &= (2 \times \pi \times r) \times (r + h).\end{aligned}$$

Problems

13. A room 15 feet wide and 20 feet long and 10 feet high is to be painted. How many square feet of wall space must be covered? What is the total area of the room?
14. The radius of a tin can is 2 inches and the height is 3 inches. What is the circumference of the base? What is the volume of the can? What is the total surface area of the can?

We can consider the surface of any pyramid by making a model, cutting away the base, and then making one cut along a lateral edge in order to flatten out the lateral surface on the plane. This gives us a polygon and a series of triangles. By measuring the bases and heights of these figures we can compute their areas and add them up for the total area.

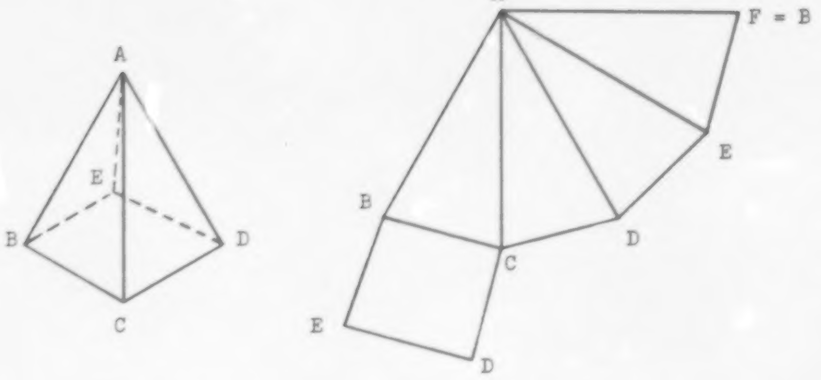


Figure 28-17. The surface of a pyramid.

The only type of cone which can be treated easily in the same manner is the special one whose base is a circle and whose vertex lies exactly over the center of the base.

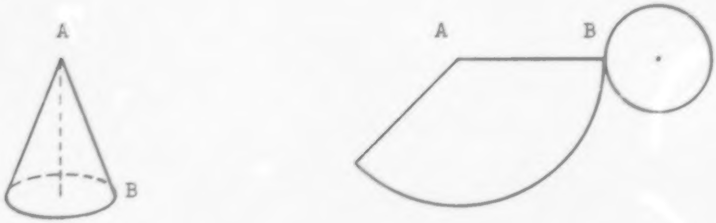


Figure 28-18. Surface area of a right circular cone.

In this case the lateral area is a fraction of the circle whose radius is congruent to an element of the cone as shown in Figure 28-18.

The area of a sphere is harder to get since no cut will ever enable anyone to flatten out the surface of a sphere into a plane. This is the reason maps of the earth printed on a flat page can never be completely accurate. However, it is interesting to know that the area of the sphere in Figure 28-13c is in fact exactly equal to the lateral area of the cylinder in Figure 28-13b. This enables us to write the formula

$$\begin{aligned}\text{Area of a sphere} &= C \times 2 \times r\\ &= (2 \times \pi \times r) \times (2 \times r)\\ &= 4 \times \pi \times r \times r\end{aligned}$$

where r is the radius of the sphere and C is the circumference of a circle whose radius is r.

Exercises - Chapter 28

1. Suppose the base of a right rectangular prism is left unchanged and the measure of its lateral edge doubled, what is the effect on the volume?

2. Suppose ℓ and w of a right rectangular prism are each doubled and the lateral edge left unchanged, what is the effect on the volume?

3. If each of ℓ, w, h is doubled for a rectangular prism, what is the effect on the volume?

4. Is a lateral edge of a right prism an altitude of the prism? Why?

5. Is a lateral edge of an oblique prism an altitude of the prism? Why?

6. In finding the volume of an oblique prism, a student accidentally used the length of a lateral edge in place of the height of the prism. If he made no other errors, was his answer too large or too small?

7. If the altitude of a prism is doubled, its base unaltered and all angles unchanged, how does this affect the volume?

8. If all edges of a rectangular prism are doubled and its shape left unchanged, how is the volume affected?

9. The side of the square base of a pyramid is doubled. The height of the pyramid is halved. How is the volume affected?

10. The sides of a rectangular prism are all doubled. How is the total area affected?

11. If the radius of a circle to the nearest inch is 3 inches, the circumference is 19 inches. Use this fact to find the volume of a cylinder whose radius is 6 inches and height is 12 inches.

12. Use the information in Exercise 11 to find the volume of a cone whose radius is 6 inches and height is 12 inches.

13. Use the information in Exercise 11 to find the volume of a sphere of radius 6 inches.

14. Use the information in Exercise 11 to find the surface area of the cylinder in Exercise 11 and of the sphere in Exercise 13.

15. Use the appropriate value of π to find the volume and area of the spheres with radii listed below.
   a. r is 3 inches. b. r is 4000 miles. c. r is .01 inch.

Solutions for Problems

1. $V = (5 \times 3) \times 6 = 90$. Answer: 90 cubic inches

2. $V = (5.2 \times 3.4) \times 6.3 = 111.384$. Since the sides of the prism are measured in units of .1 inch the unit of volume is .001 cubic inch. So our answer in terms of this unit of volume is 111,384 cubic thousandths of an inch. We express the answer in cubic inches as 111.384 cubic inches. The question of whether or when we should round off such an answer is a very tricky one which we will not attempt to discuss here. If the answer is desired to the nearest cubic inch, it would be 111 cu. in., if to the nearest .1 of a cubic inch, it would be 111.4 cu. in.

   We do not know whether this answer is less than or more than the exact value of V. If the sides of the prism were measured to the nearest .01 inch, they might well come out such numbers as 5.16, 3.37 and 6.26 or 5.24, 3.43, 6.34. In the first case our answer would be greater than the next better approximation to V, in the second case it would be less.

3. $V = B \times h$, $B = \frac{1}{2} \times 6 \times 8 = 24$, $V = 24 \times 12 = 288$
   Answer: 288 cubic inches

4. $V = B \times h = 12 \times 3 = 36$ Answer: 36 cubic inches

5. $l = 6$ ft. $l = 2$ yds. $w = 3$ yds. $h = \frac{5}{3}$ yds.
   $V = 2 \times 3 \times \frac{5}{3} = 10$ Volume is 10 cu. yds. Answer: This is a 10 ton truck.

6. $V = B \times h$. $B = \pi \times r \times r = \pi \times 12 \times 12$. $V = \pi \times 12 \times 12 \times 35$

$$= \frac{22}{7} \times 5040 = 22 \times 720$$

$$= 15,840$$

   Answer: 15,840 cubic inches. We may use $\frac{22}{7}$ for π since the measurements are to the nearest inch.

7. $V = \frac{1}{3} \times B \times h = \frac{1}{3} \times 12 \times 12 \times 10 = 480$. Answer: 480 cubic feet

8. The Columbia Encyclopedia gives the dimensions as base 768 feet by 768 feet and height 482 feet.

$$V = \frac{1}{3} \times 482 \times 768 \times 768$$

   Answer: 31,585,942 cubic feet

9. The cylinder must have height 4 feet in order for it to have the same volume as the given cone.

10. $V = \frac{1}{3} \times B \times h = \frac{1}{3} \times \pi \times 3 \times 3 \times 10 = 30 \times \pi = 30 \times \frac{22}{7} = \frac{660}{7}$

$V = 94.3$ Answer: 94 cubic inches

11. $V = \frac{4}{3} \times \pi \times 3 \times 3 \times 3 = 36 \times \pi = 36 \times \frac{22}{7} = \frac{792}{7}$

$V = 113.1$ Answer: 113 cubic inches

12. $V = \frac{4}{3} \times \pi \times .1 \times .1 \times .1 = \frac{.004}{3} \times \pi = \frac{.004}{3} \times 3.1416 = \frac{.0125664}{3}$

$V = .004188$ Answer: .004 cubic inches

13. Wall space is the lateral area. This $= 10 \times [15 + 20 + 15 + 20]$

$= 10 \times 70 = 700$.

Wall space is 700 square feet.

Total area $= 700 + 2 \times 300 = 700 + 600 = 1300$. Answer: 1300 square feet

14. Circumference of base is $2 \times \pi \times r = 4 \times \pi = \frac{88}{7} = 12.6$. Answer: 13 inches

$V = \pi \times 2 \times 2 \times 3 = 12 \times \frac{22}{7} = \frac{264}{7} = 37.7$. Answer: 38 cubic inches

Total surface area $= (2 \times \pi \times r) \times (r + h)$

$= (2 \times \pi \times 2) \times (2 + 3)$

$= 4 \times \pi \times 5 = 20\pi = 20 \times \frac{22}{7} = \frac{440}{7}$

$= 63$

Answer: 63 square inches

Chapter 29

NEGATIVE RATIONAL NUMBERS

Introduction

Up to now, three different sets of numbers have been studied. They are:

the counting numbers: $1, 2, 3, \ldots,$

the whole numbers: $0, 1, 2, 3, \ldots,$

the rational numbers: $0, \ldots, \frac{1}{2}, \ldots, \frac{9}{3}, \ldots, \frac{29}{7}, \ldots$.

In this chapter we will extend once again the concept of number by adding the notion of oppositeness or direction to the idea of number which we already have. This will result in a new set of numbers, the set of positive, negative and zero rational numbers. From now on this new set of numbers will be called the rational numbers and the third set above will be called the non-negative rationals. If 0 is omitted from this set, the new set so obtained is called the positive rationals.

Each of the three systems of numbers mentioned above arose historically out of a practical need and was used to express certain qualities and properties of a physical model. Our new set of numbers also arose out of a physical situation which needed a mathematical interpretation.

This was a situation where counting or measuring was with respect to a fixed reference point from which the direction of counting or measuring was important. Examples of such situations are measuring temperature in degrees, or altitude in feet. One talks about a temperature of 33.5 degrees above zero or 33.5 degrees below zero, an altitude of 300 feet above or 300 feet below sea level. A business firm may have a credit balance or a debit at its bank. In each case a number by itself will measure the size or the magnitude involved but without mention of the direction, full information on the physical situation is not given.

In each of these physical situations there are essentially only two directions involved. We can indicate one direction with the superscript " $^{+}$ " and the other with the superscript " $^{-}$ ". The choice as to which direction is labelled $^{+}$ and which is purely arbitrary although often the physical situation indicates the best choice. Thus, usually we indicate temperature above zero, distances to the north from a fixed starting point, altitude above sea level and a credit balance at the bank as being in the positive direction in each case. On the other hand, a deep sea diver might want to indicate the depths of his dives as being in the positive direction and the bank might want to consider a firm's credit balance as money it owes to a depositor and so as being in the negative direction as far as the bank's assets and liabilities are concerned.

A combination such as $^{+}\frac{1}{3}$ or $^{-}3$ of one of the superscripts and one of the numerals for a positive rational gives us an effective name for a number which may indicate both direction and size. Such a number is a rational number. Actually $^{+}\frac{1}{3}$ is merely another name for the number $\frac{1}{3}$ which is one of the positive rationals studied in Chapters 17-24. For these positive rationals we will use or omit the superscript $^{+}$ as happens to be convenient and instructive at the moment. $^{-}3$ is, however, a name for a new kind of number. It is a negative rational number, one of the whole set of such numbers which we are now introducing and are going to study in this chapter. Together with 0 and the positive rationals, these form the set of rational numbers. Sometimes we call these the signed rationals or just the signed numbers when we want to emphasize the directions involved. $^{-}2$, $^{-}\frac{1}{2}$, $^{+}1$ are names which are read "negative two," "negative one-half" and "positive one."

Using these signed numbers we can say, having agreed that temperatures above zero are considered to be in the positive direction, "The boiling point of water is $^{+}212^{\circ}$" or "The elevation of a town is $^{+}600$ feet" if it is above sea level and we have chosen this as the positive direction.

Since from now on we will be talking mostly about rational numbers, we will use the simple "numbers" as short for "rational numbers." If any other system is intended, we will say so.

A Physical Model

When we studied whole numbers and positive rational numbers, we found that a good physical model such as a number line was a great help to our understanding. Once again a number line will serve our purpose admirably. We start as before by picking a point to represent 0 which we call the origin and label 0. We then pick a direction on the line for the positive direction and a unit length. The point one unit from 0 in the positive direction we label 1, or $^{+}1$, the point 2 units in the positive direction from 0 we label 2. In general, any point which on the number line in Chapter 18 was labelled with the number $\frac{a}{b}$ may now also be labelled $^{+}\frac{a}{b}$. See Figure 29-1.

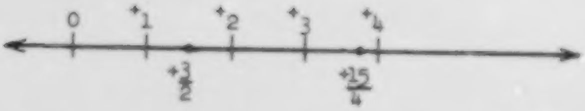


Figure 29-1. Positive rational numbers on the number line.

So far the points of our number line which are on one of the rays from 0 represent only the positive rational numbers and 0. But now, using the ray in the opposite direction from 0, and calling it the negative direction we can represent the negative rational numbers by points on it. Thus, we label the point one unit in the negative direction from 0 with $^-1$, the point two units in the negative direction, $^-2$, the point halfway between, $^-\frac{3}{2}$, etc.

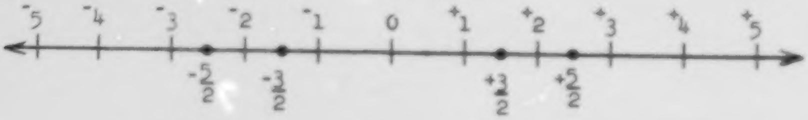


Figure 29-2. Rational numbers on the number line.

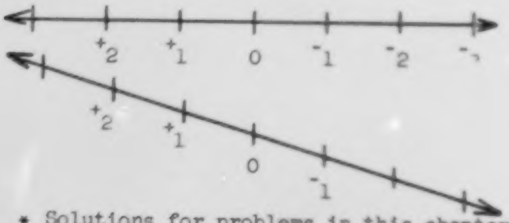
When using the number line, we sometimes talk loosely about the "point" or the "number" rather than the "point representing the number." We hope this will not confuse you. It will certainly save time and effort.

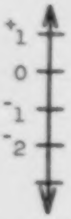
Figure 29-2 makes it apparent that the positive numbers may be thought of an extending indefinitely to the right of 0, and the negative numbers indefinitely to the left of 0.

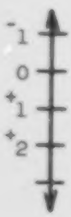
Pairs of numbers such as $^-1$ and $^+1$, $^+\frac{3}{2}$ and $^-\frac{3}{2}$ are said to be "opposites." Thus $^-2$ is the opposite of $^+2$ and $^+\frac{3}{2}$ is the opposite of $^-\frac{3}{2}$. Zero is considered to be its own opposite. Pairs of opposite numbers such as $^-4$ and $^+4$ are represented by pairs of points the same distance to the right and left of 0. We write $^-4$ = opp (4) and 4 = opp ($^-4$), etc.

Problems*

1. Are the following possible number line models of rational numbers?







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* Solutions for problems in this chapter are on page 406.

2. a. What is the opposite of $^{-}3$?
   b. What is the opposite of the opposite of $^{-}3$?
   c. What is the opposite of 0?

Arrows: A Second Model

We come now to the question of operations on signed numbers. Can we add, subtract, multiply and divide them and, if we can, do the operations have the properties of closure, commutativity and associativity? Is multiplication distributive over addition?

We already know about the non-negative rationals since they have been studied extensively. The real questions are: can we extend the definitions of addition, multiplication, etc. from these numbers to all rational numbers so that all the familiar properties still hold? The answer is yes. To help us see how, we construct a second physical model of rational numbers.

In our previous chapters we used several different physical models for whole numbers and rational numbers to help illuminate different mathematical characteristics. Up to now a number, say 3, has been represented by a point on the number line. Our new model for the number 3 consists of a directed line segment instead of a point. See Figure 29-3. We represent this directed segment by an arrow which points in the direction from the point 0 to the point 3 and whose length is 3 units. If the positive direction of the number line is from left to right, the model for 3 will be an arrow 3 units long pointing to the right. Similarly, the model for $^{-}\frac{3}{2}$ is an arrow one and a half units long pointing to the left. In general if the number r has been represented in our first model by the point R on the number line, the new model is an arrow pointing in the direction from 0 to R and whose length is equal to the measure of $\overline{OR}$. Notice that we have not said where the arrow begins, just how long it is and in which direction it points. Either arrow in Figure 29-3 may represent $^{+}3$.

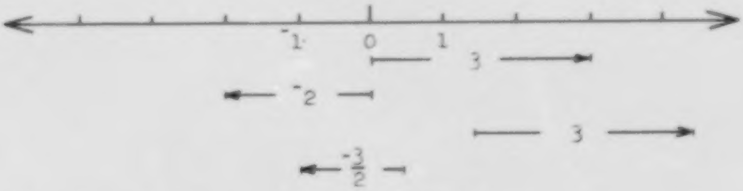


Figure 29-3. Directed arrows as models of rational numbers.

Addition of Rationals

This new model will help us greatly in thinking about addition of numbers. Remember that an operation on two numbers associates a third number with an ordered pair of numbers. What number do we associate with $^{+}5$ and $^{+}3$ as their sum? Since these are just new names for 5 and 3, we know that their sum is 8. Thus, $^{+}5 + {}^{+}3 = {}^{+}8$. If we thought of $^{+}5$ as representing $5.00 earned one day and $^{+}3$ as $3.00 earned a second day, the sum $^{+}8$ would tell us we were $8.00 better off at the end of the second day than at the beginning of the first. But if we had lost $3.00 the second day, we would be only $2.00 better off. We could represent this situation as $^{+}5 + {}^{-}3$ and the answer seems to be $^{+}2$. This sum is easily seen using our arrows as models to represent numbers. In our model, addition of two numbers is represented by drawing an arrow representing the first and then from the head of this arrow drawing an arrow representing the second. The sum is then represented by the arrow which goes from the tail of the first to the head of the second. Thus in Figure 29-4 the arrows represent $^{+}5 + {}^{+}3 = {}^{+}8$, while in Figure 29-5 the arrows represent $^{+}5 + {}^{-}3 = {}^{+}2$.

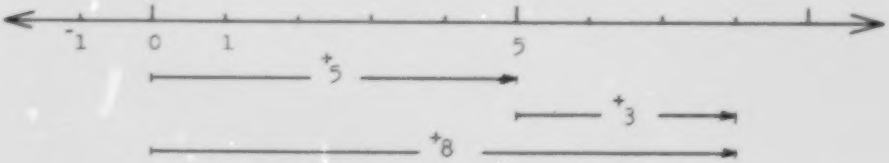


Figure 29-4. Addition of two positive rationals.

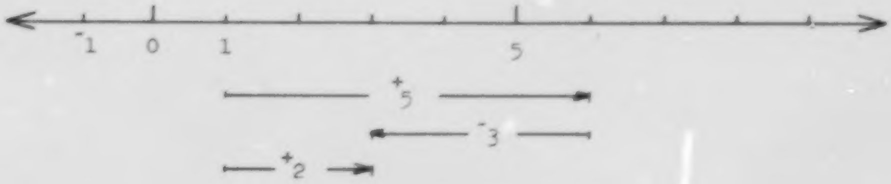


Figure 29-5. Addition of a positive and a negative number.

In these models the arrows should all be drawn in the same line but the figure would be hard to follow, so we have drawn them in separate lines. Also notice that in Figure 29-4 the arrow for $^{+}5$ started opposite the point 0 on the number line while in Figure 29-5 the arrow started opposite the point 1. It is important to realize that in this model for addition the first arrow may start anywhere on the number line. Actually, the number line is important in the figures only to indicate the positive direction and the scale of our model. It is not where the arrow starts but its direction and length which tell us what number it represents. Thus in Figure 29-5 the sum of $^{+}5$ and $^{-}3$ is an arrow which goes from 1 to 3 and is therefore 2 units long and headed in the positive direction. It represents $^{+}2$. In later figures we will draw just enough of the number line to indicate the positive direction and the scale.

Thus, Figure 29-6 is a model for $^{+}3 + {}^{-}5$.

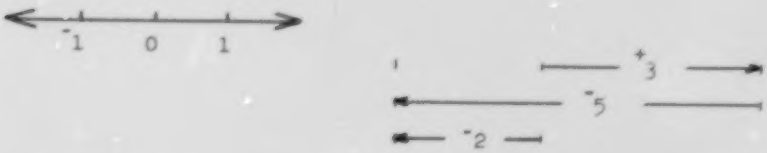


Figure 29-6. $^{+}3 + {}^{-}5 = {}^{-}2$.

The sum is modeled by an arrow 2 units long and directed to the left. The sum is, therefore, $^{-}2$. Figure 29-7 shows the addition of two opposite numbers, $^{+}2$ and $^{-}2$. The sum is 0 and so it will always be for any two opposite numbers.

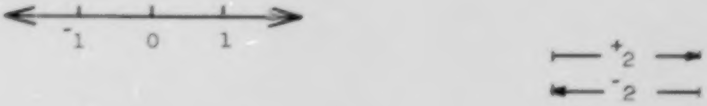


Figure 29-7. $^{+}2 + {}^{-}2 = 0$.

We can equally well add two negative numbers and see the answer using arrow models. See Figure 29-8. In this case our number line is drawn as a vertical line.

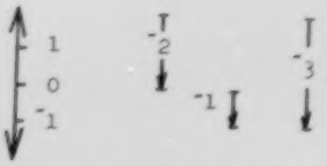


Figure 29-8. $^{-}2 + {}^{-}1 = {}^{-}3$.

Problems

3. Draw a model to represent ${}^{+}3 + {}^{-}4$. Also for ${}^{-}3 + {}^{+}4$ and ${}^{-}3 + {}^{+}3$.
4. Draw a model to represent ${}^{-}2 + {}^{+}3$. Also for ${}^{+}3 + {}^{-}5$ and ${}^{+}1 + {}^{-}6$.
5. Draw a model to represent $\frac{+3}{2} + \frac{+3}{2}$. Also $\frac{+5}{2} + \frac{-5}{2}$ and $\frac{-9}{5} + 4$.
6. Draw a model to represent ${}^{-}3 + {}^{-}4$. Also for ${}^{-}2 + {}^{-}3$.

The operation of addition may become clearer if you think of it as moving from home in a certain direction a given distance and then following this with another walk in the same or the opposite direction for another given distance and finally determining where you are with respect to the starting point. Thus, a walk $3\frac{3}{4}$ blocks east followed by one $5\frac{1}{2}$ blocks west brings you to a point $1\frac{3}{4}$ blocks west of your starting point. $3\frac{3}{4} + 5\frac{1}{2} = {}^{-}1\frac{3}{4}$. The arrows in Figure 29-9 show you starting at a point 3 blocks east of City Hall and ending at the point $1\frac{3}{4}$ west of where you started, that is, at a point $1\frac{1}{4}$ blocks east of City Hall.

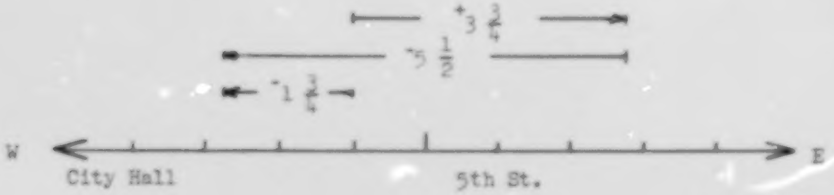


Figure 29-9.

Certain general features of the model of addition presented here should be noted.

1. In an addition $b + c = s$ where b and c denote numbers and s the unknown sum, the arrow for the first addend b is always drawn first.
2. The arrow for the second addend c is drawn next. Its tail starts at the head of the arrow for b and its direction depends on whether the number c is positive or negative.
3. The arrow giving the unknown sum s is then drawn. Its tail is always in line with the tail of the arrow for b and its head in line with the head of the arrow for c. The length of this arrow and its direction determines s.
4. If the arrow for b is drawn with its tail at 0 the sum s is always the number on the number line opposite the head of the sum arrow.

From our model it can also be observed that addition of rational numbers is commutative and associative. Figure 29-10 illustrates commutativity.

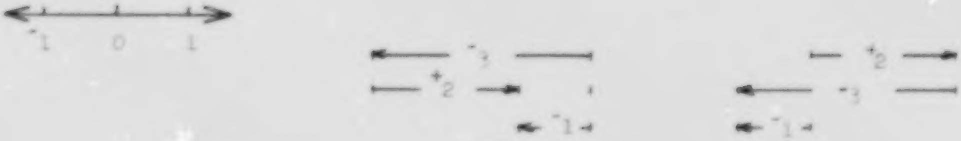


Figure 29-10. Commutativity of addition for rationals: $^-3 + {}^+2 = {}^+2 + {}^-3$.

Although this one example does not prove the property, it is in fact true.

Associativity may be shown by examples, such as

$$({}^-3 + {}^-5) + {}^+4 = {}^-8 + {}^+4 = {}^-4$$

and also

$$^-3 + ({}^-5 + {}^+4) = {}^-3 + {}^-1 = {}^-4\ .$$

This property in fact holds for the addition of any three rationals.

Problems

7. Draw models to show that $^-3 + {}^-2 = {}^-2 + {}^-3$.
8. Draw models to show that $({}^-3 + {}^+2) + {}^-1 = {}^-3 + ({}^+2 + {}^-1)$.
9. Draw a model to show that $\frac{^-2}{3} + \frac{^-4}{3} = {}^-2$.

Order of the Rational Numbers

Let us return to the number line model for rationals and let A name the point which corresponds to the number a etc. When using the number line for positive rational numbers, we saw that if $a > b$, then on a number line in which 1 lay to the right of 0, point A lay to the right of point B. Also, if point C lay to the right of point D then $c > d$.

We say exactly the same thing for the rational numbers and their number line. If 1 is to the right of 0 then 2 is to the right of $^-4$, $^-3$ to the right of $^-10$ and we say 2 is greater than $^-4$, $^-3$ is greater than $^-10$, or in mathematical symbols, $2 > {}^-4$, $^-3 > {}^-10$. We may also say $^-4$ is less than 2 and $^-10$ is less than $^-3$, writing $^-4 < 2$ and $^-10 < {}^-3$.

Another way of saying that 5 is to the right of 2 is to say that we must add a positive number to 2 to get 5. In the same way we must add a positive number to $^-10$ to get $^-3$. Hence, $5 > 2$ and $^-3 > {}^-10$ mean $5 = 2 +$ some positive number and $^-3 = {}^-10$ plus some positive number.

In fact, if $\underline{r}$ and $\underline{s}$ represent any two rational numbers and if $r + p = s$ where $\underline{p}$ is a positive number, then $s > r$. Also if $s > r$, then there must always be a positive number $\underline{p}$ such that $r + p = s$.

Problems

10. Which is correct: ${}^{-}2 > {}^{-}3$ or ${}^{-}3 > {}^{-}2$? Show the solution on the number line and also by producing the positive number called for in the second definition.

11. Label each as True or False.

a. ${}^{-}2 > 0$	e. $\frac{-7}{8} > \frac{-8}{9}$
b. ${}^{-}3 > {}^{+}3$	f. $\frac{7}{8} > \frac{8}{9}$
c. ${}^{+}5 > {}^{-}2$	g. $\frac{-7}{8} > \frac{+8}{9}$
d. $\frac{-2}{3} > \frac{-3}{4}$	

Subtraction of Rationals

We have considered addition of rationals very carefully. What can we say about subtraction? We can consider the subtraction of $\underline{r}$ and $\underline{s}$ in two ways: first directly via the physical models, the arrows which represent $\underline{r}$ and $\underline{s}$; and second as the inverse of addition, but again using the arrows to represent the proper addition problem. In the first case we have to consider what sort of combination of the two arrows is a model for subtraction; in the second case we say that $r - s$ is the number which answers the question, "What number added to $\underline{s}$ will give $\underline{r}$?" It is important, of course, that both methods are consistent, i.e., give the same answer. In the first case consider ${}^{+}5 - {}^{+}2$. Since ${}^{+}5$ and ${}^{+}2$ are simply different names for the numbers 5 and 2 we know the answer must be ${}^{+}3$. What could we do to the arrow representing ${}^{+}2$ so that we could combine it with the one for ${}^{+}5$ and get the one for ${}^{+}3$? The answer is quickly seen to be "Reverse the direction of the ${}^{+}2$ arrow and add it to the ${}^{+}5$ arrow." But reversing the direction of a ${}^{+}2$ arrow simply gives us an arrow representing the opposite of ${}^{+}2$ or ${}^{-}2$. This gives us the hint we need. Our physical model of the subtraction of a rational number will be to reverse its arrow and add. Thus:

$${}^{+}5 - {}^{+}2 = {}^{+}5 + {}^{-}2 = {}^{+}3$$

and
$${}^{+}2 - {}^{+}5 = {}^{+}2 + {}^{-}5 = {}^{-}3$$

and
$${}^{+}2 - {}^{-}5 = {}^{+}2 + {}^{+}5 = {}^{+}7$$

Actually making physical models of these arrows and using them in this fashion is an extremely useful and illuminating experiment. Again

$$^{-}7 - {}^{-}15 = {}^{-}7 + {}^{+}15 = {}^{+}8$$

and in general for any two rational numbers r and s

$$r - s = r + \text{opposite of } s.$$

The second look at subtraction is as the inverse of addition. Thus $^{+}4 - {}^{+}7$ asks us for a number n, if any, such that $^{+}7 + n = {}^{+}4$. But the arrow model shows that there is an arrow from the head of $^{+}7$ to the head of $^{+}4$ and clearly this arrow by size and direction represents $^{-}3$.

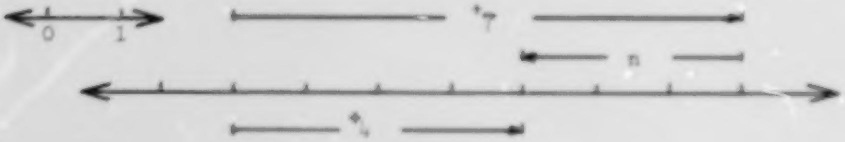


Figure 29-11. If $^{+}7 + n = {}^{+}4$, $n = {}^{-}3$.

We also see that: if $^{+}4 - {}^{+}7 = n$ then

$^{+}4 = {}^{+}7 + n$ but this

$= n + {}^{+}7$ and if we add $^{-}7$ we get

$^{+}4 + {}^{-}7 = (n + {}^{+}7) + {}^{-}7$ which

$= n + ({}^{+}7 + {}^{-}7)$ or

$= n + 0$

$= n.$

So we have $^{+}4 + {}^{+}7 = {}^{+}4 - {}^{-}7$ and once again we see that the subtraction of any rational number may be replaced by the addition of its opposite. Unlike the systems we have had before, the rational numbers are closed under subtraction, i.e., for any two rational numbers, p and q, there always exists a unique rational number, s, such that $p - q = s$. This was not true for any of the systems we have studied before. There was no whole number n such that $n = 4 - 7$, nor rational number r such that $r = \frac{4}{5} - \frac{17}{3}$.

Problems

12. Complete the sentences.

a. $7 - {}^{-}2 =$

b. $3 - 5 =$

c. ${}^{-}3 - {}^{-}5 =$

d. $\frac{-2}{3} - \frac{2}{3} =$

e. $\frac{4}{3} - \frac{-4}{3} =$

f. $\frac{-4}{5} - \frac{-4}{5} =$

g. ${}^{+}3 - {}^{-}3 =$

13. Label as True or False.

a. ${}^{+}7 - {}^{-}3 = {}^{+}7 + {}^{-}3$

b. ${}^{+}7 - {}^{-}3 = {}^{+}7 + {}^{+}3$

c. ${}^{+}7 - {}^{-}3 = {}^{-}7 + {}^{-}3$

d. ${}^{+}7 + {}^{-}3 = {}^{-}7 - {}^{+}3$

Our rational number system is closed under subtraction, but suddenly we find we don't need subtraction any more as any subtraction can be replaced by an addition of the opposite number. This parallels closely what happened when we examined division for the positive rationals. We found that division by any number except zero could be replaced by multiplication by its reciprocal. In some ways enlarging the number system pays off in increased simplicity of operation as well as in applicability to many new kinds of physical situations.

Multiplication of Rationals

What about multiplication of rational numbers? Since the signed numerals such as ${}^{+}7$ and ${}^{+}5$ are simply different names for the positive rational numbers 7 and 5, their product is already known. Thus we may write

$${}^{+}7 \times {}^{+}5 = 7 \times 5 = 35 = {}^{+}35$$

and

$$\frac{+1}{3} \times \frac{+2}{5} = \frac{1}{3} \times \frac{2}{5} = \frac{2}{15} = \frac{+2}{15}$$

The product ${}^{+}3 \times {}^{-}2$ may be thought of as corresponding to the situation that if we lose \$2.00 a day for three days, we have lost a total of \$6.00. Mathematically, the number sentence would be ${}^{+}3 \times {}^{-}2 = {}^{-}6$. We hope multiplication of signed numbers is going to be commutative and therefore that ${}^{-}2 \times {}^{+}3$ will also be ${}^{-}6$. But physically this might be thought of as corresponding to the question, "If I am making \$3.00 a day now, how did my financial situation 2 days ago compare to it today?" Obviously, I was \$6.00 poorer. And it seems true that ${}^{-}2 \times {}^{+}3 = {}^{-}6$. On the other hand, if you are spending $3.00 a day, how did your situation compare 2 days ago with it today? Obviously, you were $6.00 better off. This corresponds to ${}^-2 \times {}^-3 = {}^+6$.

Another way to consider multiplication of positive and negative numbers is to consider the pattern in the following sequence.

We know ${}^+2 \times {}^+3 = {}^+6$

and ${}^+1 \times {}^+3 = {}^+3$

and $0 \times {}^+3 = 0.$

Now what should ${}^-1 \times {}^+3$, ${}^-2 \times {}^+3$ be?

If the pattern of dropping 3 units each time on the right side of these equations is to be preserved, the answers should be ${}^-3$ and ${}^-6$ respectively.

Thus ${}^-1 \times {}^+3 = {}^-3$

and ${}^-2 \times {}^+3 = {}^-6.$

If the commutative property is to hold, we should also have

$$ {}^+3 \times {}^-2 = {}^-6. $$

But then another pattern will develop as we see that

$$ 2 \times {}^-2 = {}^-4 $$

$$ 1 \times {}^-2 = {}^-2 $$

$$ 0 \times {}^-2 = 0. $$

Now it seems right that

$$ {}^-1 \times {}^-2 = {}^+2 $$

and $${}^-2 \times {}^-2 = {}^+4.$$

Corresponding to the physical model and these patterns, we can define multiplication of rational numbers as follows. Suppose r and s represent positive rationals. We already know from Chapter 21 that $r \times s$ is a positive rational. We define

$$ r \times {}^-s = {}^-(r \times s); $$

$$ {}^-r \times s = {}^-(r \times s); $$

$$ {}^-r \times {}^-s = {}^+(r \times s) = r \times s; $$

and finally $${}^-r \times 0 = 0 \times {}^-r = 0.$$

Thus, $^-2 \times {}^-5 = {}^+(2 \times 5) = {}^+10$ and $^-\frac{1}{2} \times {}^+\frac{2}{3} = {}^-(\frac{1}{2} \times \frac{2}{3}) = {}^-\frac{1}{3}$

and $^+3 \times {}^-\frac{1}{3} = {}^-1$ and $0 \times {}^-\frac{2}{3} = 0.$

Does multiplication thus defined distribute over addition? Let's look at an example.

$$^-5 \times (^+6 + {}^-2) = {}^-5 \times {}^+4 = {}^-20$$

and

$$(^-5 \times {}^+6) + (^-5 \times {}^-2) = {}^-30 + {}^+10 = {}^-20.$$

So in this case it does distribute. We will not attempt a general proof of this or other properties, but will assure you that indeed multiplication for the signed numbers is closed, commutative, associative and distributive over addition. The familiar properties of 0 and 1 also hold.

A Second Look at Multiplication

Another way of approaching the whole question of multiplication of rational numbers is to assume at the beginning that multiplication and addition of any two rational numbers do have all the properties which these operations have for the non-negative numbers as studied in Chapters 18-21. Then, assuming this is true, we want to show that the product of a negative number and a positive number is a negative number and that the product of two negative numbers is a positive number. To do this we will need only a few of the properties, notably the commutative property of multiplication, the property that multiplication is distributive over addition and the property that zero times any number is zero.

The distinctive property of the set of rational numbers beyond the set of non-negative rationals is that in the set of rationals there always exists for any number a a unique opposite number b such that $a + b = 0$. For example: the opposite of 5 is $^-5$ and $5 + {}^-5 = 0$; the opposite of $^-\frac{3}{5}$ is $\frac{3}{5}$ and $^-\frac{3}{5} + \frac{3}{5} = 0$; the opposite of 0 is 0 and $0 + 0 = 0$.

Suppose, now, we want to determine what rational number is equal to $2 \times {}^-3$. In other words, since $2 \times {}^-3$ is a numeral for some number, what is the simplest numeral for this number?

Consider $2 \times {}^-3$. If we add 2×3 to it we get $(2 \times {}^-3) + (2 \times 3)$. By using the distributive property, the property of opposites and the property of zero, we get

$$\begin{aligned}(2 \times {}^-3) + (2 \times 3) &= 2 \times (^-3 + 3) \\ &= 2 \times 0 \\ &= 0.\end{aligned}$$

Therefore, $2 \times {}^{-}3$ is the opposite of 2×3. Since $2 \times 3 = 6$, its opposite, $2 \times {}^{-}3$, must be ${}^{-}6$ or ${}^{-}(2 \times 3)$. In short,

$$2 \times {}^{-}3 = {}^{-}(2 \times 3).$$

By the commutative property

$${}^{-}3 \times 2 = 2 \times {}^{-}3 = {}^{-}(2 \times 3) = {}^{-}(3 \times 2) = {}^{-}6.$$

Now consider ${}^{-}3 \times {}^{-}2$. We add ${}^{-}3 \times 2$ to it and get again

$$\begin{aligned}({}^{-}3 \times {}^{-}2) + ({}^{-}3 \times 2) &= {}^{-}3 \times ({}^{-}2 + 2) \\ &= {}^{-}3 \times 0 \\ &= 0.\end{aligned}$$

Therefore, ${}^{-}3 \times {}^{-}2$ is the opposite of ${}^{-}3 \times 2$. Since ${}^{-}3 \times 2 = {}^{-}6$, its opposite must be 6. In short,

$${}^{-}3 \times {}^{-}2 = 6 = 3 \times 2.$$

Thus, in this instance the product of a negative number by a positive one is negative and the product of a negative number by a negative one is positive. But we can follow exactly the same pattern for any numbers.

Suppose a and b represent any two positive numbers, then we know $a \times b$ is positive and each of ${}^{-}a$, ${}^{-}b$ and ${}^{-}(a \times b)$ is a negative number and is the unique opposite of the corresponding positive number. Consider $a \times {}^{-}b$ and add $a \times b$. We get

$$\begin{aligned}(a \times {}^{-}b) + (a \times b) &= a \times ({}^{-}b + b) \\ &= a \times 0 \\ &= 0.\end{aligned}$$

Therefore $(a \times {}^{-}b)$ is the opposite of $a \times b$ and must be just another name for ${}^{-}(a \times b)$ since there is only one opposite for any number.

By the commutative property

$$({}^{-}a \times b) = b \times {}^{-}a = {}^{-}(b \times a) = {}^{-}(a \times b).$$

Consider $({}^{-}a \times {}^{-}b)$ and add ${}^{-}a \times b$. We get

$$\begin{aligned}({}^{-}a \times {}^{-}b) + ({}^{-}a \times b) &= {}^{-}a \times ({}^{-}b + b) \\ &= {}^{-}a \times 0 \\ &= 0.\end{aligned}$$

Therefore, $({}^{-}a \times {}^{-}b)$ is the unique opposite of ${}^{-}a \times b$. But we just proved that this is $a \times b$. So

$$({}^{-}a \times {}^{-}b) = a \times b.$$

Division

Division by any rational number r $(r \neq 0)$ is, as before, defined as the inverse of multiplication by r. Thus, ${}^{+}8 \div {}^{-}4 = {}^{-}2$ since ${}^{-}4 \times {}^{-}2 = {}^{+}8$ and ${}^{-}\frac{3}{5} \div {}^{-}\frac{3}{4} = {}^{+}\frac{4}{5}$ since ${}^{-}\frac{3}{4} \times {}^{+}\frac{4}{5} = {}^{-}\frac{3}{5}$. In each case $r \div s$ is that number which multiplied by s will produce r. We see that the same rules for determining the sign of the answer apply as in multiplication. Also, if the reciprocal of r is that number $\frac{1}{r}$ which multiplied by r produces 1, it can be seen that division by any rational number $\neq 0$ is equivalent to multiplying by its reciprocal. Thus ${}^{+}3 \div {}^{-}5 = {}^{+}3 \times {}^{-}\frac{1}{5} = {}^{-}\frac{3}{5}$.

Problem

14. Complete.

a. ${}^{+}\frac{2}{3} \times {}^{-}\frac{3}{5} =$

b. ${}^{-}\frac{2}{3} \div {}^{+}\frac{2}{3} =$

c. ${}^{-}3 \div {}^{+}\frac{1}{3} =$

d. ${}^{-}3 \times {}^{-}\frac{5}{2} =$

e. ${}^{-}\frac{3}{5} \div {}^{-}\frac{3}{5} =$

f. ${}^{+}\frac{2}{3} \div {}^{-}\frac{2}{3} =$

g. ${}^{+}\frac{2}{3} \times 0 =$

h. ${}^{-}\frac{5}{2} \div {}^{-}\frac{2}{5} =$

Summary

In the rational number system subtraction of r can be replaced by addition of the opposite of r and division by $r, \neq 0$, by multiplication by the reciprocal of r. All the rules of the operations have been explored, but not all the properties. It has been asserted that the standard properties of closure, commutativity and associativity do hold for addition and multiplication. The rational numbers are also closed under subtraction and under division by all non-zero numbers. This provides a number system capable of describing mathematically many more physical situations than we have been able to handle up to now.

In the next chapter we will make one more extension of the number system in order to name all the points on the number line with numbers. Did you think that because the rational points were dense (see Chapter 19) on the number line there were no unnamed points? If so, you are wrong. In Chapter 30 we will find a point which does not correspond to any rational number.

Exercises - Chapter 29

1. Copy and complete the following sentences by writing the correct sign between the numbers, " > ," "< ," or " = ."

a. $^{+}3 \underline{\qquad} ^{+}5$

b. $\frac{-12}{17} \underline{\qquad} \frac{-4}{5}$

c. $\frac{-8}{3} \underline{\qquad} \frac{+6}{4}$

d. $^{+}1 \underline{\qquad} ^{-}19$

e. $^{-}16 \underline{\qquad} ^{-}32$

f. $^{+}479 \underline{\qquad} ^{+}421$

g. $^{+}89 \underline{\qquad} ^{+}95$

h. $^{-}26 \underline{\qquad} ^{-}26$

i. $\frac{-3}{7} \underline{\qquad} \frac{-5}{12}$

j. $0 \underline{\qquad} ^{-}7$

2. Write the number that is

a. 3 greater than $^{-}12$

b. 7 less than 0.

c. $\frac{4}{5}$ greater than $\frac{+16}{5}$

d. 5 less than $\frac{-2}{3}$

e. 6 greater than 0.

f. 2 less than $^{+}9$.

3. Which arrow has the greater measure?

a. $^{-}2$ to $^{+}6$ or $^{+}2$ to $^{+}6$

b. $^{+}8$ to $^{+}1$ or $^{+}8$ to $^{-}1$

c. 0 to $^{+}4$ or $^{-}6$ to 0

d. $^{+}5$ to $^{-}3$ or $^{-}3$ to $^{-}5$

e. $^{-}4$ to $^{-}8$ or $^{+}6$ to $^{+}9$

4. Complete each equation:

a. $^{-}4 + ^{-}3 = \underline{\qquad}$

b. $\frac{+2}{3} - \frac{-4}{5} = \underline{\qquad}$

c. $^{+}3 - ^{+}5 = \underline{\qquad}$

d. $^{+}4 + ^{-}9 = \underline{\qquad}$

e. $^{-}7 + ^{+}14 = \underline{\qquad}$

f. $^{+}8 - ^{-}3 = \underline{\qquad}$

g. $^{-}11 + ^{+}10 = \underline{\qquad}$

h. $\frac{-9}{4} + \frac{-7}{4} = \underline{\qquad}$

5. Complete these mathematical sentences:

a. $^{+}8 + \underline{\qquad} = ^{-}15$

b. $\underline{\qquad} - ^{-}9 = ^{-}15$

c. $^{+}20 - \underline{\qquad} = ^{-}15$

d. $^{+}1 + \underline{\qquad} = ^{-}15$

e. $^{-}2 + \underline{\qquad} = ^{+}6$

f. $\underline{\qquad} + ^{+}15 = ^{+}6$

g. $\underline{\qquad} + ^{+}3 = ^{+}6$

h. $^{+}9 + \underline{\qquad} = ^{+}6$

6. Which statements are true about 0?
   a. It is neither positive nor negative.
   b. It is equal to its opposite.
   c. It is less than any negative number.
   d. It is the sum of any number and its opposite.
   e. It changes the value of any number to which it is added.
   f. It is less than any positive number.

7. Complete: Use "positive" or "negative" to complete these sentences:
   a. If a number is greater than its opposite, the number is a ____ number.
   b. If a number is less than its opposite, the number is a ____ number.
   c. When you add two negative numbers, the sum is a ____ number.
   d. When you add two positive numbers, the sum is a ____ number.
   e. When you add a negative number and a positive number, the sum is a ____ number if the positive addend is further away from 0 than is the negative addend.
   f. When you add a positive number and a negative number, the sum is a ____ number if the negative addend is further away from 0 than is the positive addend.

8. Complete these mathematical sentences. Show that the order of adding two addends may be changed without changing the sum.
   a. ${}^{+}7 + {}^{-}4 =$ ____
   b. ____ $= {}^{-}12 + {}^{-}6$
   c. ${}^{+}3 + {}^{+}11 =$ ____
   d. ____ $= {}^{-}16 + {}^{-}7$
   e. ____ $= {}^{+}8 + {}^{-}13$
   f. ${}^{-}6 + {}^{-}9 =$ ____
   g. ____ $= {}^{-}5 + {}^{+}10$
   h. ${}^{+}32 + {}^{-}19 =$ ____

9. Complete the mathematical sentences with " > ," " < " or " = ."
   a. ${}^{-}3 + {}^{-}6$ ____ ${}^{-}6 + {}^{-}3$
   b. ${}^{+}3 + {}^{-}6$ ____ ${}^{-}3 + {}^{+}6$
   c. ${}^{+}6 + {}^{+}3$ ____ ${}^{-}3 + {}^{-}6$
   d. ${}^{-}6 + {}^{+}3$ ____ ${}^{+}3 + {}^{-}6$
   e. ${}^{+}7 + {}^{-}2$ ____ ${}^{+}7 + {}^{+}2$
   f. ${}^{+}2 + {}^{-}7$ ____ ${}^{-}7 + {}^{+}2$
   g. ${}^{-}2 + {}^{-}7$ ____ ${}^{-}2 + {}^{-}7$
   h. ${}^{-}2 + {}^{+}7$ ____ ${}^{-}7 + {}^{+}2$

10. If $^-6 + {}^+2$ is written in each blank below, will the sentence be True or False?

a. ____ $> {}^-8 + {}^-8$

b. ____ $< 0 + {}^+5$

c. ____ $> {}^+6 - {}^+2$

d. ____ $> {}^+6 + {}^-2$

e. ____ $< {}^-4 + {}^-2$

f. ____ $< 0 + {}^-6$

g. ____ $> {}^+5 + {}^-10$

h. ____ $< {}^-7 - {}^-9$

11. Complete the following mathematical sentences:

a. $^-3 \times {}^+4 =$

b. $\frac{^-2}{3} \times \frac{^-3}{5} =$

c. $\frac{^-2}{5} \times \frac{^-5}{2} =$

d. $\frac{^-3}{5} \times \frac{5}{3} =$

e. $0 \times \frac{^-2}{3} =$

f. $^-6 \times ({}^-2 \times {}^-3) =$

g. $(\frac{^-2}{3} \times \frac{4}{5}) \times \frac{^-3}{2} =$

h. $(\frac{5}{1} \times \frac{^-1}{5}) \times {}^-1 =$

12. Compute:

a. $\frac{^+6}{5} + \frac{^-2}{3}$

b. $\frac{^-3}{4} + \frac{^+4}{3}$

c. $\frac{^-3}{5} + \frac{^-3}{5}$

d. $0 + \frac{^+2}{3}$

e. $\frac{^+4}{5} + 0$

f. $(\frac{^+2}{3} + \frac{^-3}{3}) + \frac{^-1}{3}$

g. $({}^-3 - {}^-3) + {}^+3$

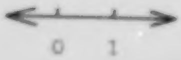
h. $({}^+5 - {}^-3) + {}^-2$

Solutions for Problems

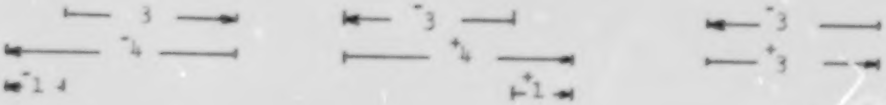
1. Yes. Each are perfectly good number lines for representing rational numbers. The crucial items are a choice of 0 and a choice of direction for the ray on which the positive rationals are to be represented. This ray may go in either direction and then, of course, the negatives must be represented on the opposite ray.

2. a. 3
   b. $^-3$ since the opposite of $^-3$ is 3 and the opposite of 3 is $^-3$.
   c. 0. 0 is its own opposite.

The following scale is to be used in the answers to Problems 3-8.



3.



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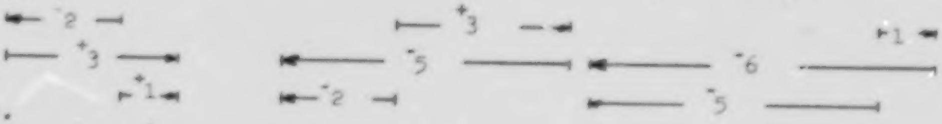

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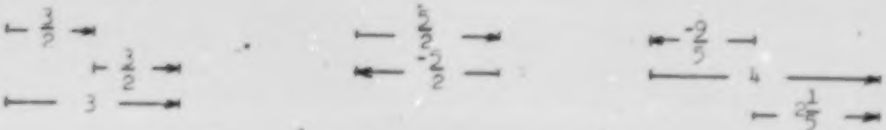
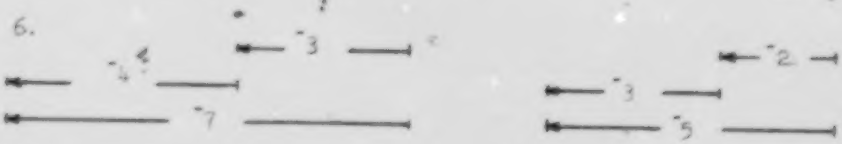
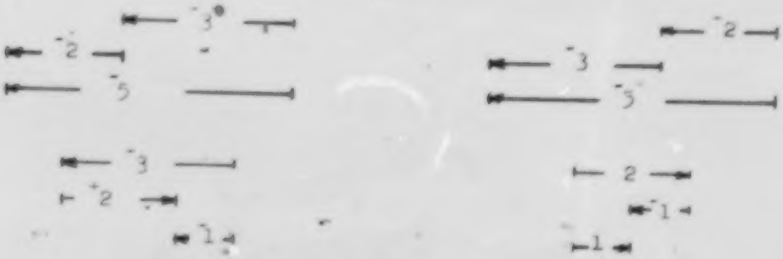

$\frac{3}{2}$
$\frac{3}{2}$
3
$\frac{5}{2}$
$\frac{^{-}2}{2}$
$\frac{^{-}2}{5}$
4
$2\frac{1}{5}$

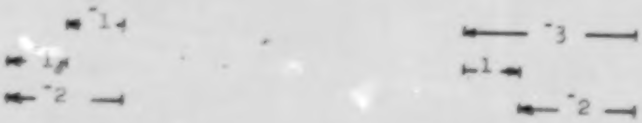
6.



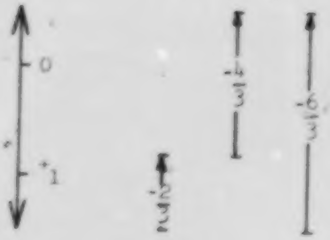
7.


$^{-}3$
$^{-}2$
$^{-}5$
$^{-}2$
$^{-}3$
$^{-}5$
$^{-}3$
$^{+}2$
$^{-}1$
2
$^{-}1$
1

8.



9.


0
$^{+}1$
$\frac{^{-}1}{2}$
$\frac{^{-}4}{3}$
$\frac{^{-}6}{3}$

10. $^{-}2 > {}^{-}3$,

$^{+}1$ is to the right of 0 and $^{-}2$ is to the right of $^{-}3$. Also $^{-}3 + {}^{+}1 = {}^{-}2$ and $^{+}1$ is the positive number required by the definition.

11. a. False

b. False

c. True

d. True

e. True

f. False

g. False

12. a. $7 - {}^{-}2 = 7 + 2 = 9$

b. $3 - 5 = 3 + {}^{-}5 = {}^{-}2$

c. $^{-}3 - {}^{-}5 = {}^{-}3 + 5 = 2$

d. $\frac{^{-}2}{3} - \frac{2}{3} = \frac{^{-}2}{3} + \frac{^{-}2}{3} = \frac{^{-}4}{3}$

e. $\frac{4}{3} - \frac{^{-}4}{3} = \frac{4}{3} + \frac{4}{3} = \frac{8}{3}$

f. $\frac{^{-}4}{5} - \frac{^{-}4}{5} = \frac{^{-}4}{5} + \frac{^{+}4}{5} = 0$

g. $^{+}3 - {}^{-}3 = {}^{+}3 + {}^{+}3 = {}^{+}6$

13. a. **False:** $^{+}7 - {}^{-}3 = {}^{+}7 + {}^{+}3$

b. True

c. False (see a)

d. **False:** $^{+}7 + {}^{-}3 = {}^{+}7 - {}^{+}3$

14. a. $\frac{^{+}2}{3} \times \frac{^{-}3}{5} = \frac{^{-}6}{15} = \frac{^{-}2}{5}$

b. $\frac{^{-}2}{3} \div \frac{^{+}2}{3} = \frac{^{-}2}{3} \times \frac{3}{2} = {}^{-}1$

c. $^{-}3 \div \frac{^{+}1}{3} = {}^{-}3 \times \frac{^{+}3}{1} = {}^{-}9$

d. $^{-}3 \times \frac{^{-}5}{2} = \frac{^{+}15}{2}$

e. $\frac{^{-}3}{5} \div \frac{^{-}3}{5} = \frac{^{-}3}{5} \times \frac{^{-}5}{3} = {}^{+}1$

f. $\frac{^{+}2}{3} \div \frac{^{-}2}{3} = \frac{^{+}2}{3} \times \frac{^{-}3}{2} = {}^{-}1$

g. $\frac{^{+}2}{3} \times 0 = 0$

h. $\frac{^{-}5}{2} \div \frac{^{-}2}{5} = \frac{^{-}5}{2} \times \frac{^{-}5}{2} = \frac{^{+}25}{4}$

Chapter 30

THE REAL NUMBERS

Introduction

As was said in Chapter 29, we are now ready to make the last extension (for us) of the number system. In the first twelve chapters our concern was the whole numbers, their operations and properties. In Chapters 18-24 the system of non-negative rational numbers was developed and the operations on these numbers and the properties of the operations were studied. At the time we called this set of numbers the rational numbers although, more accurately, we should have called them, as we did just now, the non-negative rationals. In Chapter 29 we developed the complete system of rational numbers including the negative numbers and studied their operations and properties. Remember that now "rational number" refers to any such number as:

$$2, \quad 3, \quad \frac{^-2}{3}, \quad 0, \quad 1.7, \quad {^-2.34}, \quad \frac{^-15}{3}, \quad 17, \quad \text{etc.}$$

From now on we shall almost always write the positive rationals omitting the + superscript. When a letter such as a or r is used to represent a rational number it should be understood that it may represent 0 or a negative rational, just as well as a positive one. We found that the processes of computing for the negative rationals--the operations of addition, subtraction, multiplication, and division--and their properties such as commutativity and associativity are consistent with the processes and properties for the positive rationals. This is important, of course, if one is to avoid having to learn completely new techniques each time the number system is extended.

Properties of Operations on the Rational Numbers

Let us summarize the properties of the system of rational numbers:

1. Closure Properties: If a and b are rational numbers, then $a + b$ and $a \times b$ are rational numbers.
2. Commutative Properties: If a and b are rational numbers, then $a + b = b + a$ and $a \times b = b \times a$.
3. Associative Properties: If a, b and c are rational numbers, then $(a + b) + c = a + (b + c)$ and $(a \times b) \times c = a \times (b \times c)$.
4. Identities: There is a rational number zero, 0, such that if a is a rational number, then $a + 0 = 0 + a = a$. There is a rational number one, 1, such that if a is a rational number $a \times 1 = 1 \times a = a$.

5. Distributive Property: If a, b and c are rational numbers, then $a \times (b + c) = (a \times b) + (a \times c)$.
6. Reciprocals: If r is a rational number, not zero, then there is another rational number s, such that $r \times s = 1$. Every rational number except zero has a reciprocal. Often s is written as $\frac{1}{r}$ and then $r \times \frac{1}{r} = 1$.
7. Opposites: If r is any rational number, then there is a rational number t, such that $r + t = 0$. Usually t is written as ^-r and $r + {}^-r = 0$.

Properties 6 and 7 enable us to convert any division and subtraction problems into multiplication and addition problems. Hence, they make it true that the rational numbers are closed for subtraction and closed for division by any number except 0.

8. Order: If r and s are any two distinct rational numbers, the statement $r < s$ is the same statement as: a *positive* rational number p can be found such that $r + p = s$, (which can also be written as $s - r = p$).

In Chapter 19 it was shown that if any two rational numbers are given, there is always another rational number between them. Another way of saying this is that if r is a rational number, there is no next larger one. This property of rational numbers is called *density*. We say that the rational numbers are dense and include the property in our list.

9. Density: Between any two distinct rational numbers there is at least a third rational number.

From this it follows that there are many rational numbers and corresponding to them on the number line, many rational points. Moreover, the points are spread throughout the number line. Any segment, no matter how small, contains infinitely many rational points. One might think that all the points on the number line are rational points, that is, that every point on the line corresponds to a rational number. *This is not so*. There are many points on the line that are not assiciated with rational numbers. This is shown in the section *An Irrational Point*.

Irrational Numbers

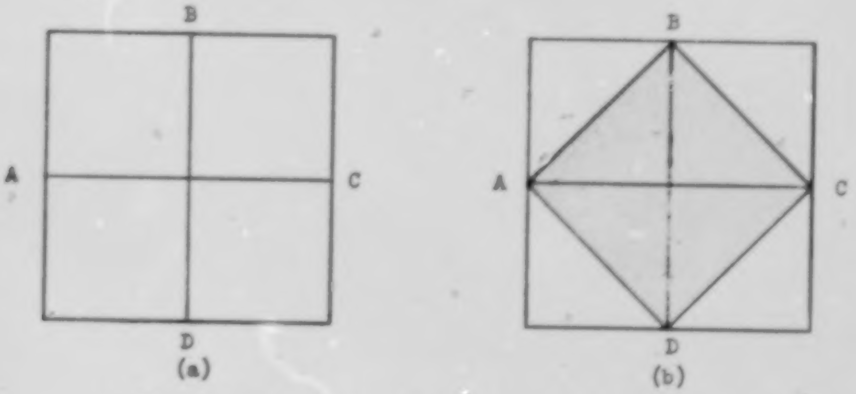
If these points whose existence we have promised to show are not associated with rational numbers, are they associated with any numbers at all? Obviously not, unless somehow we can extend our number system again, and this is precisely what we now do. We simply say:

> To every point on the number line not associated with a rational number we associate a new kind of number which we will call an irrational number.

Irrational numbers are not the product of an irrational mind. "Rational" has two quite different meanings and "irrational" means in each case simply "not rational." When referring to numbers, it means just that the number is one which cannot be expressed as a fraction with an integer as numerator and a counting number as denominator. Note that a fraction is one way of expressing a "ratio" and this is the origin of the word "rational."

An Irrational Point

It was said above that not every point on the line corresponds to a rational number. With the rational numbers being dense this might seem like a pretty strong statement. Can we prove it? Can we find even one point on the number line which definitely does not correspond to a rational number? Look at a floor tiled in the pattern illustrated in Figure 30-1. If the small squares are 1 foot on a side, the area of each small square in Figure 30-1a is 1 × 1 or 1 square foot and a total of 4 square feet for the whole figure. Dividing each small square in half by a diagonal segment yields the shaded square in Figure 30-1b whose area is 2 square feet.

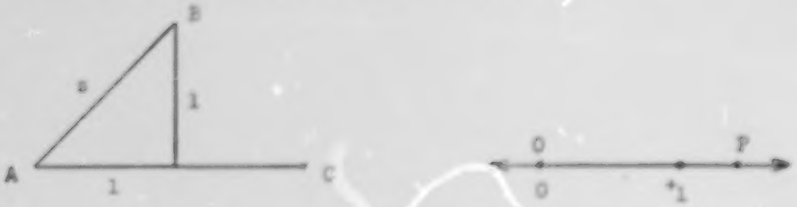


(a) (b)

Figure 30-1. Four tiles each one foot square.

If the length of the side of this square is s, it must be true, by the area formula for a rectangle in Chapter 27, that $s \times s = 2$.

We lay off a segment congruent to $\overline{AB}$ from O on a number line to a point P. The number associated with P should be s.



$\overline{OP} \cong \overline{AB}$, $m(\overline{OP}) = s$.

Figure 30-2.

If we can show that s is not a rational number, we have found a point which is not associated with a rational number. This will show that, in spite of the density of the rational numbers and the points associated with them, there are points of the number line left over. The numbers associated with these points are the irrational numbers.

How then can it be proved that s is not a rational number?

Indirect Reasoning

To do this, a line of reasoning which people very often use will be followed. This type of reasoning can be illustrated by the following conversation between a mother and her son. John was late from school. When his mother scolded him, he tried to avoid punishment by saying that he had run all the way home. "No, you didn't run all the way," she said firmly. John was surprised and ashamed and asked, "How did you know?" "If you had run all that way, you would have been out of breath," she said. "You are not out of breath. Therefore, you did not run."

John's mother had used indirect reasoning. She assumed the opposite of the statement she wished to prove, and showed that this assumption led to a conclusion which was not true. Therefore, her assumption had to be false, and the original statement had to be true.

Proof that s is not Rational

To prove that s is not a rational number, indirect reasoning will be used. It will be assumed that s is a rational number and then it will be shown that this assumption leads to an impossible conclusion.

Assume that s is a rational number. Then s can be written as $\frac{p}{q}$, where p and q are integers and $q \neq 0$. Take $\frac{p}{q}$ in simplest form. This means that p and q have no common factor except 1.

Since $s \times s = 2$, we know that $\frac{p}{q} \times \frac{p}{q} = 2$.

Thus, $\frac{p \times p}{q \times q} = \frac{2}{1}$ which means $(p \times p) \times 1 = 2 \times (q \times q)$.

Thus $p \times p$ is an even integer. Now p itself is either even or odd. But the product of two even numbers is always even and the product of two odd numbers is always odd. For example, $6 \times 8 = 48$ and $4 \times 4 = 16$ while $5 \times 7 = 35$ and $3 \times 3 = 9$. So if p were odd, $p \times p$ would be odd and we know it is even. Therefore p must be an even number which we can write as $2 \times m$. From this it follows that

$$\begin{aligned} 2 \times (q \times q) &= p \times p \\ &= (2 \times m) \times (2 \times m) \\ &= 2 \times [m \times (2 \times m)] \\ &= 2 \times [2 \times (m \times m)] \end{aligned}$$

Therefore, dividing by 2: $q \times q = 2 \times (m \times m)$ which shows that $q \times q$ is an even number. But this means that q is also an even number and can be written as $2 \times n$. Therefore, both p and q have the common factor 2 contradicting the fact with which we started, i.e., that they have no common factor except 1.

So the statement "s is a rational number" must be false. That is, s is not a rational number, but the point P in Figure 30-2 is a perfectly definite point on the number line.

We now have a point on the number line and no rational number to associate with it. We simply assert that there is an irrational number s associated with this point. Since $s \times s = 2$ and the radical sign $\sqrt{}$ is used for square roots, we give the name $\sqrt{2}$ to this number.

In similar fashion, we can prove that there are points on the number line such that if there were numbers b and c associated with them, it would have to be true that $b \times b = 3$ and $c \times c = 5$, but we can also prove that b $(= \sqrt{3})$ and c $(= \sqrt{5})$ are not rational numbers. We assert that there are irrational numbers corresponding to these points. Furthermore, the product of any rational number by an irrational number is irrational so we have many, many irrational numbers. Some further examples are $\sqrt{3} + \sqrt{2}$, π, $\frac{\sqrt{2}}{3}$, etc.

We have said that to every point on the number line not associated with a rational number there corresponds an irrational number. It is also true that to each irrational number there corresponds a point on the number line which does not correspond to any rational number. The rational numbers although dense on the number line leave many points unnamed. The irrational numbers fill all the gaps.

The question may well be raised: How can the point on the number line corresponding to a given irrational number be found? We know from Chapter 23 that every rational number can be written as a terminating or repeating decimal and that every terminating or repeating decimal is a numeral for a rational number. We now assert that every non-repeating decimal is a numeral for an irrational number and that every irrational number may be written as a non-repeating decimal. To locate the point corresponding to any number we express that number as a decimal. For example, $\pi = 3.14159\ldots$ where we have written six digits and indicated by the dots the fact that there are more to come. The points 3 and 4 are located on the number line. The point for π, if there is one, lies between these points, since $\pi > 3$ and $4 > \pi$. It also lies between the points 3.1 and 3.2, between the points 3.14 and 3.15, between 3.141 and 3.142, 3.1415 and 3.1416, 3.14159 and 3.14160, etc. Thus we can locate the position of the point with as great accuracy as we please and we assert that there actually is a point which corresponds to the number π and to no other number.

The subject of irrational numbers is a very tricky and subtle one. We have made many statements and assertions about them, and will make a few more, which we have not proved at all, but we are doing this deliberately so that you can compare them with the rationals.

Real Numbers

The union of the set of all the rational numbers and the set of all the irrational numbers is called the set of real numbers. Every point of the number line has been associated either with a rational or an irrational number. So every point of the number line is associated with a real number and hence the number line is called the real number line.

We thus have a new extended system of numbers, the real numbers, but how do we add, subtract, multiply and divide them? Do the familiar properties hold here as well as for the rationals? As for the first question we shall show in the next paragraph how to add and multiply any two real numbers. Subtraction and division are as usual defined in terms of addition and multiplication. To answer the second question we state that the familiar properties of these operations do indeed hold, but the explanations and justifications are beyond the scope of this course.

We know that every real number can be expressed as a decimal, terminating or repeating if the real number is rational, non-repeating and hence non-terminating if it is irrational. We can find the sum or product of any two real numbers to as great a degree of accuracy as we wish by writing them as decimals, writing for a non-terminating decimal as many places as may be required. In computing the sum or product we make use of the properties that if $\underline{a}$, $\underline{b}$, $\underline{c}$ and $\underline{d}$ are positive numbers with $a < b$ and $c < d$ then $a + c < b + d$, and $a \times c < b \times d$. Also, if $a \times a < b \times b$ then $a < b$. We have not proved these properties, but you might check them with a few examples.

Then $1.41 < \sqrt{2} < 1.42$

$1.73 < \sqrt{3} < 1.74.$

Therefore: $3.14 < \sqrt{2} + \sqrt{3} < 3.16$

and $2.4393 < \sqrt{2} \times \sqrt{3} < 2.4708.$

Thus, if we want to add or multiply the two real numbers $\sqrt{2}$ and $\sqrt{3}$ the first two digits of the sum are 3.1 and of the product 2.4. Note that if accuracy to more than one decimal place is wanted in either the sum or the product of $\sqrt{2}$ and $\sqrt{3}$, it will be necessary to write $\sqrt{2}$ and $\sqrt{3}$ to three or more decimal places.

Summary and a Look Ahead

We have in a certain sense completed our development of number systems since we now have a number system which is in one-to-one correspondence with all the points on a number line. The numbers are ordered according to their position on the number line. They can be added, subtracted, multiplied and divided (except by zero) and the resulting numbers are on the number line. We find the properties listed at the beginning of this unit all hold for the real numbers as well as for the rationals.

In fact, you might suspect that we are getting such a close relationship between numbers and points that we should be able to study one by means of the other. It is as though arithmetic, the study of numbers and their relationships, and geometry, the study of sets of points and their relationships, could profitably be considered together. This is true and the discovery of this fact was one of the great achievements of 17th century mathematics. The idea is to relate more closely the study of sets of numbers and the study of sets of points. This brings together two of the streams of mathematical concepts we have been looking at, those of arithmetic and those of geometry. If the arithmetic and geometric concepts are tied together with the concepts of measurement they furnish some very powerful tools for the further study of physical situations and of mathematical relationships.

In high school and college students will continue to study numbers and geometric figures. The numbers will be the real numbers, the figures those of plane and solid geometry, the union of these two will lead to analytic geometry, calculus and many fields of higher mathematics. The foundation for all this later work must be laid by the study of the arithmetic of the whole and rational numbers and the geometry of simple figures.

Exercises - Chapter 30

1. Which of the following are irrational numbers?
   a. $\sqrt{2}$
   b. $\frac{3 - \sqrt{2}}{5 + 2}$
   c. 0.12
   d. $\frac{16}{111}$
   e. $9 - \sqrt{3}$
   f. $\sqrt{5} + \sqrt{2}$
   g. $\sqrt{7}$
   h. $\frac{8}{9}$

2. Is π rational or irrational?

3. What is the reciprocal of $\sqrt{2}$? Is it irrational?

4. Is $\frac{\sqrt{3}}{3}$ the reciprocal of $\sqrt{3}$?

5. Is the product of two irrational numbers always an irrational number? Why or why not?

6. What is the opposite of each of the following?
   - a. $\sqrt{2}$
   - b. $\frac{1}{\sqrt{2}}$
   - c. $\sqrt{3} - \sqrt{2}$
   - d. $\sqrt{5} + \sqrt{2}$
   - e. $\sqrt{7}$
   - f. $^{-}\sqrt{5}$

7. Is the sum of two irrational numbers always an irrational number? Why or why not?

8. A rule for writing a non-repeating decimal may be given as follows: Start by writing 1.1, then write a 0 getting 1.10, next write two 1's, getting 1.1011, then a zero followed by three 1's, and continue in this fashion. Thus, to fourteen decimals we have 1.10110111011110... . Is this non-repeating decimal a rational or an irrational number?

9. The number whose decimal expansion is obtained by writing 0.123,456,789,101,112,131, ... which is all the numerals written in order is again a non-repeating decimal. Is it rational or irrational? Why?

10. The sum of a rational and an irrational number is always irrational. To show this we can use indirect reasoning.

    If r is rational and s is irrational and $r + s = t$, can t be rational? No, because $s = t - r$ and if t and r are both rational s would have to be rational since the rational numbers are closed under subtraction.

    Use the same type of reasoning to show that the product of a non-zero rational number and any irrational number is always irrational.

11. If the radius of a circle is 3 inches long, is the measure of the circumference a rational number? Why?

12. If the radius of a circle is 3 inches long, is the measure of the area a rational number? Why?

EPILOGUE

There are three major aspects of any study of mathematics. The first of them is the conceptual aspect which is concerned with what is being studied. What are numbers, how are they combined, what properties do they and the combination operations have? The second is the computational and manipulative aspect which is concerned with how the operations work, namely the techniques involved in adding or dividing; the algorithms that give the results quickly and easily and the skills that are necessary for accurate computing and checking.

The third aspect is that of the applications to problems of the physical world. This is concerned with one of the reasons why we study arithmetic. How is arithmetic applied to measurement, how are the problems of the world around us translated into number relations?

The focus of this book has been primarily on the first aspect, the concepts of numbers and of sets of points. This is not because it is any more important than the second or third aspect, but because in the past it has been rather badly neglected. A careful study of the fundamental concepts of number gives more meaning and more unity to the study of arithmetic. Skill in manipulation is, however, definitely important, as is the ability to apply and use mathematics.

Any study of mathematics at any level ideally should bring all its aspects in balance. We hope that this course has given you a better knowledge of the conceptual side of elementary mathematics so that you will be able to give it its rightful share in the elementary school curriculum. We also hope that you have developed some interest in numbers and sets of points, how many different kinds there are, and how they are related. There are still further systems of numbers and more complicated geometric figures which mathematicians have studied, which are interesting in their own right and which are also useful in describing more complicated physical situations.

Mathematics almost always plays a two-fold role, a tool for the sciences and an interesting, fascinating study in its own right.

ANSWERS TO EXERCISES

Chapter 2

1. a and c

2. B.

3. The elements of A and C match.

4. a. 0; b. 2; c. 5; d. 1; e. 10.

5. a and d, possibly c.

6. {⊙, □, △} = A; {a, b, c, d, e, f} = B, $N(A) < N(B)$

7. $N(A) = 3$, $N(B) = 6$.

8. Take a group of wide and narrow objects of the same kind. Put the wide ones in a set by themselves and reject the narrow ones. Repeat until the idea gets across.

9. $N(C) < N(B)$ $N(B) < N(A)$ $N(C) < N(A)$

10. picture, word, gesture, etc.

11. 0

12. It is greater than the number represented by the point if 1 is to the right of 0.

13. The greater number is represented by a point to the right if 1 is to the right of 0.

14. They indicate that the number line can be extended in either direction.

Chapter 3

1. a, b, c and d are all names for the number 3.

2. a. 5, V, 𝍸, 6-1, 3 + 2, 4 + 1

   b. 8, VIII, $\begin{matrix}|||\\|||\\||\end{matrix}$, 6 + 2, 10 - 2, 16 ÷ 2

   c. 4, ||||, IV, 8 ÷ 2, 2 × 2, 2 + 2.

3. a. 200,106; b. 2051; c. 1360; d. 1,020,253

4. II means 2, 11 means 10 + 1 or eleven.

5. a, c, e, g, h.

6. a. no; b. no; c. yes; d. yes; e. no

7. a. 5683; b. 3965; c. 7527; d. 8556; e. 10,244

Chapter 4

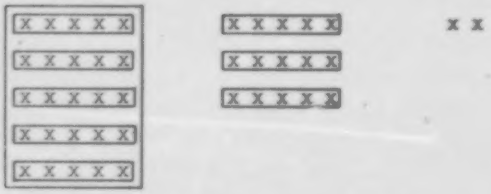
1. a. 245; b. 5; c. 1715

2. 2401

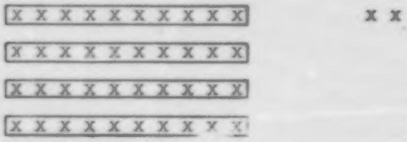
3. 132_{seven}

4. a. 7; b. There are 7 distinct one digit symbols used; c. ⌐LL;
d. □−, □|, □|, □△, □□, □⧅, □⊠, ⧅−, ⧅|, ⧅L, ⧅△.

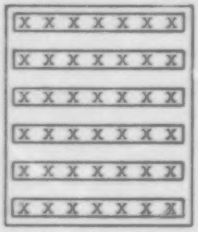
5. 0, 1, Λ, Z, 10, 11, 1Λ, 1Z, Λ0, Λ1, ΛΛ, ΛZ, Z0, Z1, ZΛ, ZZ, 100, 101, 10Λ, 10Z, 110

6. 

132_{five}

7. 

42_{ten}

8. 

60_{seven}

9. a. 0, 1, 2, 3, 4; b. 0, 1, 2, 3, 4, 5, 6; c. 0, 1, 2, 3, 4, 5, 6, 7

10. a. 14_{five} d. 30_{five}
b. 44_{five} e. 3_{five}
c. 112_{five} f. 42_{five}

11. a. 12_{seven} d. 21_{seven}
b. 33_{seven} e. 3_{seven}
c. 44_{seven} f. 31_{seven}

12. a. 13; b. 8; c. 2; d. 18; e. 24; f. 29

13. a. same d. same
b. base 3 e. base 3
c. base 3 f. base 3

14. In each part the answer is base 2.

Chapter 5

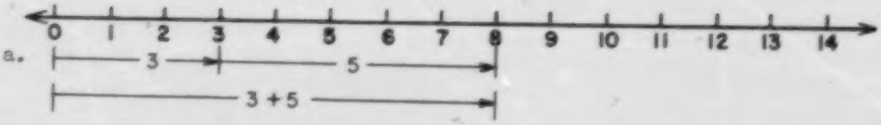
1. {dog, cat, cow, pig, duck, horse, elephant}

2. {1, 2, 3, 4, 5, 6, 7, 8, 10, 12}

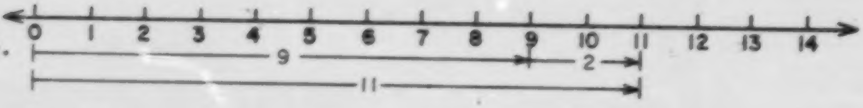
3. P

4.

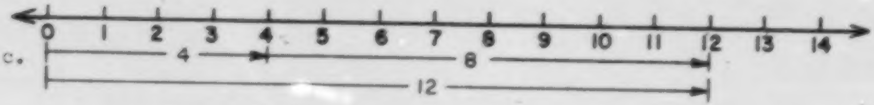
a.


0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
3
5
3 + 5

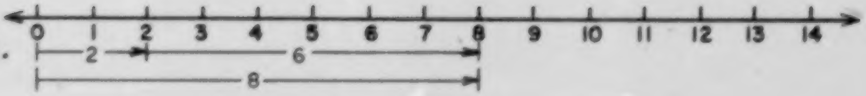
b.


0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
9
2
11

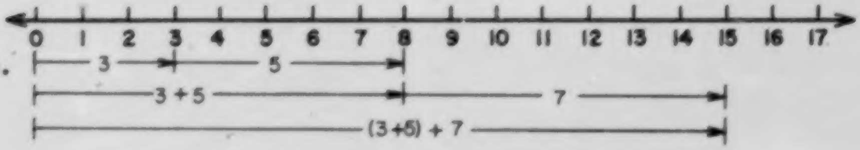
c.


0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
4
8
12

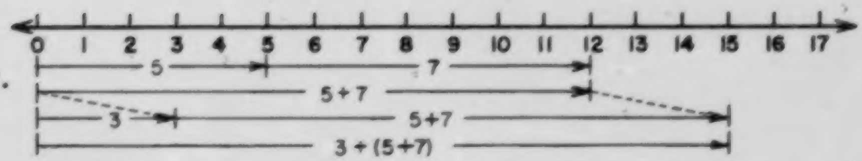
d.


0 1 2 3 4 5 6 7 8 9 10 11 12 13 14
2
6
8

e.


0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17
3
5
3 + 5
7
(3 + 5) + 7

f.


0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17
5
7
5 + 7
3
5+7
3 + (5+7)

5. a and e

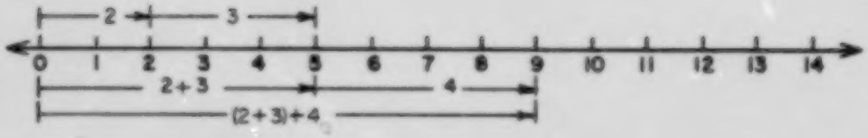
6. a. Associative
   b. Commutative
   c. Commutative
   d. Associative
   e. Commutative

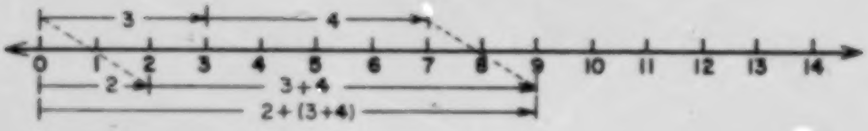
7. a. 5
   b. 0
   c. 0, 1, 2, 3, 4, 5

8. a. 1
   b. 10
   c. Any whole number
   d. Any whole number
   e. 0
   f. 0
   g. None

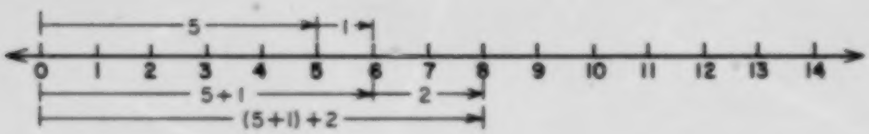
9.

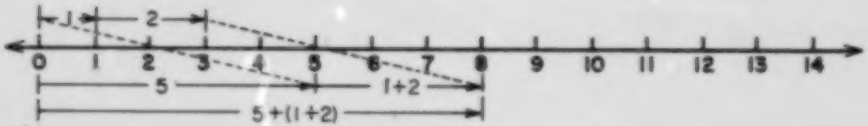
a.


2
3
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14
2+3
4
(2+3)+4


3
4
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14
2
3+4
2+(3+4)

b.


5
1
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14
5+1
2
(5+1)+2


1
2
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14
5
1+2
5+(1+2)

10. a. 4
    b. 3
    c. 7

11. The binary system is extremely simple in computation. Large numbers are tedious to write.

The duodecimal system may be used conveniently to represent large numbers. Twelve is divisible by 1, 2, 3, 4, 6 and 12, while ten is divisible only by 1, 2, 5 and 10. The twelve system requires more computational "facts" which will increase difficulties in memorizing tables of addition and multiplication. We do use twelve in counting dozens, gross, etc., and in some of the common measures of length.

12. People who work with computers often use the base eight. To change from binary to octal and back is simple with the help of the table:

Binary	Octal
000	0
001	1
010	2
011	3
100	4
101	5
110	6
111	7

For example, we have

$$2000_{ten} = 011,111,010,000_{two} = 3720_{eight}.$$

Note the grouping of numerals by threes in the binary numeral. The sum of the place values of digits in each group results in the octal numeral. Hence,

$$011 = (1 \times 2) + (1 \times 1) = 3$$
$$111 = (1 \times 4) + (1 \times 2) + (1 \times 1) = 7, \text{ etc.}$$

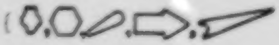
base 10	7	15	16	32	64	256
base 8	7	17	20	40	100	400
base 2	111	1111	10,000	100,000	1,000,000	100,000,000

13. Five weights; 1 oz., 2 oz., 4 oz., 8 oz., may be used to check any weight up to 15 ounces. By adding a 16 oz. weight, any weight up to 31 ounces may be checked.

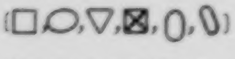
Chapter 6

1. N(A) = 3   N(B) = 4 $N(A \cup B) = 7$

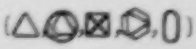
2. Subsets of A
{ }, {○,⬡,□,△}, {○}, {⬡}, {□}, {△}, {○,⬡}, {○,□},
{○,△}, {⬡,□}, {⬡,△}, {○,⬡,□}, {○,⬡,△}, {□,△},
{○,△,□}, {⬡,□,△}

3. C = {


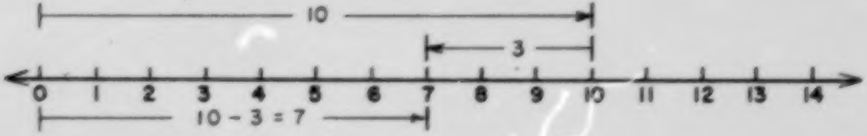
missing addend

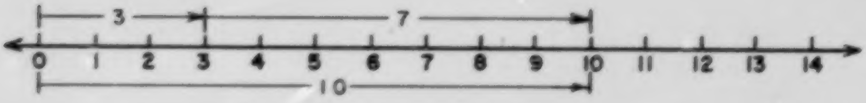
4. A ~ B =


5. 6   take away method

6. N(B) = 5, B =


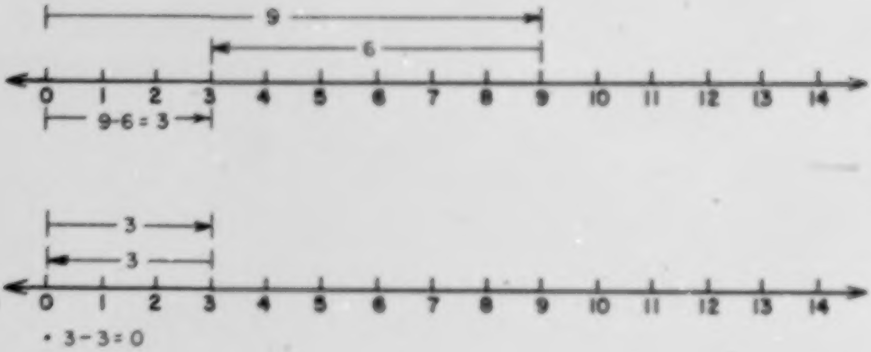
7.


10
3
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
10 - 3 = 7

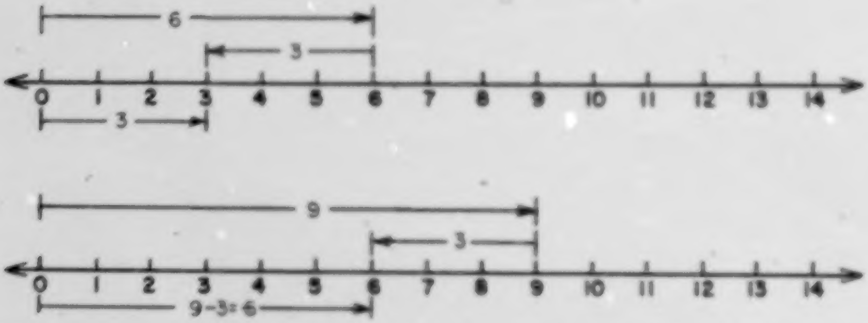

3
7
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
10

3 + 7 = 10

8.


9
6
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14
9-6=3
3
3
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14
• 3-3=0

Two number lines showing $(9 - 6) - 3 = 3 - 3 = 0$


6
3
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14
3
9
3
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14
9-3=6

Two number lines showing $9 - (6 - 3) = 9 - 3 = 6$

9. subtracting 7; adding 8

10. a. $A = \{\square, \bigcirc, \triangle\}$

$B = \{a, b\}$

$A \cup B = \{\square, \bigcirc, \triangle, a, b\}$

$(A \cup B) - B = \{\square, \bigcirc, \triangle, a, b\} - \{a, b\}$

$(A \cup B) - B = \{\square, \bigcirc, \triangle\} = A$

b. If A and B are not disjoint sets $(A \cup B) - B \neq A$

Example: $A = \{\square, \bigcirc, \triangle\}$

$B = \{\triangle, M\}$

$A \cup B = \{\square, \bigcirc, \triangle, M\}$

$(A \cup B) - B = \{\square, \bigcirc, \triangle, M\} - \{\triangle, M\}$

$(A \cup B) - B = \{\square, \bigcirc\} \neq A$

11. a, c, d

1. a.
$$\begin{array}{rcl} 246 & = & 200 + 40 + 6 \\ 139 & = & \underline{100 + 30 + 9} \\ & & 300 + 70 + 15 = 300 + 70 + (10 + 5) = 300 + (70 + 10) + 5 = 385 \end{array}$$

b.
$$\begin{array}{rcl} 784 & = & 700 + 80 + 4 \\ 926 & = & \underline{900 + 20 + 6} \\ & & 1600 + 100 + 10 = (1000 + 600) + 100 + 10 \\ & & \qquad = 1000 + (600 + 100) + 10 = 1710 \end{array}$$

c.
$$\begin{array}{rcl} 777 & = & 700 + 70 + 7 \\ 964 & = & \underline{900 + 60 + 4} \\ & & 1600 + 130 + 11 = (1000 + 600) + (100 + 30) + (10 + 1) \\ & & \qquad = 1000 + (600 + 100) + (30 + 10) + 1 = 1741 \end{array}$$

d.
$$\begin{array}{rcl} 123 & = & 100 + 20 + 3 \\ 987 & = & \underline{900 + 80 + 7} \\ & & 1000 + 100 + 10 = 1000 + 100 + 10 = 1110 \end{array}$$

e.
$$\begin{array}{rcl} 486 & = & 400 + 80 + 6 \\ 766 & = & \underline{700 + 60 + 6} \\ & & 1100 + 140 + 12 = (1000 + 100) + (100 + 40) + (10 + 2) \\ & & \qquad = 1000 + (100 + 100) + (40 + 10) + 2 = 1252 \end{array}$$

f.
$$\begin{array}{rcl} 949 & = & 900 + 40 + 9 \\ 892 & = & \underline{800 + 90 + 2} \\ & & 1700 + 130 + 11 = (1000 + 700) + (100 + 30) + (10 + 1) \\ & & \qquad = 1000 + (700 + 100) + (30 + 10) + 1 = 1841 \end{array}$$

2. 200 + 90 + 10

3. all

4. a. commutative
   b. commutative and associative

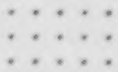
5. a.
$$\begin{array}{rcl} 764 = 700 + 60 + 4 & = & 600 + 150 + 14 \\ 199 = 100 + 90 + 9 & = & \underline{100 + 90 + 9} \\ & & 500 + 60 + 5 = 565 \end{array}$$

b.
$$\begin{array}{rcl} 402 = 400 + 0 + 2 & = & 300 + 90 + 12 \\ 139 = 100 + 30 + 9 & = & \underline{100 + 30 + 9} \\ & & 200 + 60 + 3 = 263 \end{array}$$

c.
$$\begin{array}{rcl} 710 = 700 + 10 + 0 & = & 600 + 100 + 10 \\ 287 = 200 + 80 + 7 & = & \underline{200 + 80 + 7} \\ & & 400 + 20 + 3 = 423 \end{array}$$

Chapter 8

1. a. $4 \times 5 = 20$ c. $2 \times 4 = 8$
   b. $3 \times 2 = 6$ d. $3 \times 3 = 9$

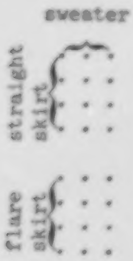
2. 

3.

top / body	red	orange	yellow	green	blue
red	▨ rr	ro	ry	rg	rb
yellow	yr	yo	▨ yy	yg	yb
blue	br	bo	by	bg	▨ bb

Total possible choices: $3 \times 5 = 15$; of these, the choices for same color top and body (shown shaded) must be ruled out. Number of choices available to Mr. Rhodes is therefore $15 - 3 = 12$.

*4. $3 \times 5 = 15$

*5. 

This is essentially the same as: the number of colors for sweater: 3, each combined with the number of colors for skirt: 4, which skirts are each available in either of 2 styles; so the total number of different ensembles is

$$5 \times 4 \times 2 = 40.$$

6. a. Always c. Never
   b. Never d. Always

7. a. Array A: $4 \times 8 = 32$;
   Array B: $4 \times 3 = 12$;
   Array C: $4 \times 5 = 20$.
   b. Yes
   c. Yes

8. $(90 \times 23) + (90 \times 27) = 2070 + 2430;$
$90 \times (23 + 27) = 90 \times 50 = 4500$
Jerry collects 4500 cents or $45.00.

9. The rows are not disjoint sets.

10. $20 \times (11 + 28 + 11) = 20 \times 50 = 1000.$
Capacity of auditorium is 1000 seats.

11.

$96 + 248 = 96 + (4 + 244)$	renaming
$= (96 + 4) + 244$	associative
$= 100 + 244$	renaming
$= 344$	renaming

Note: the label, "renaming" is included to identify what was occuring; it is not meant to state a property of the operation. This renaming may be considered a property of a number--that it may have many names.

12. a. $5 \times 4 \times 3 \times 2 \times 1 = (5 \times 2) \times (4 \times 3) \times 1$
$= 10 \times 12$
$= 120$

b. $125 \times 7 \times 3 \times 8 = (125 \times 8) \times (7 \times 3)$
$= 1000 \times 21$
$= 21,000$

c. $250 \times 14 \times 4 \times 2 = (250 \times 4) \times (14 \times 2)$
$= 1000 \times 28$
$= 28,000$

13. a. False
b. True
c. True
d. False
e. False

14. $3 \times (4 + \underline{3}) = (3 \times 4) + (3 \times 3)$
b. $2 \times (\underline{4} + 5) = (2 \times 4) + (\underline{2} \times 5)$
c. $13 \times (6 + 4) = (13 \times \underline{6}) + (13 \times \underline{4})$
d. $(2 \times 7) + (3 \times \underline{7}) = (\underline{2} + \underline{3}) \times 7$

Chapter 9

1. a. $20 \div 5 = n, \quad n = 4$ d. $72 \div 9 = n, \quad n = 8$
   b. $28 \div p = 4, \quad p = 7$ e. $64 \div 8 = n, \quad n = 8$
   c. $6 \div n = 1, \quad n = 6$ f. $42 \div q = 7, \quad q = 6$

2. a. meaningless f. meaningless
   b. 0 g. ambiguous
   c. ambiguous h. meaningless
   d. 0 i. 0
   e. 0 j. 0

3. 59 has only 1 and 59 as factors; hence, only one array is possible. 60 has many different factors; hence, many different arrays are possible.

4. No; for example, $6 \div 3 = 2$ but $3 \div 6$ is not a whole number.

5. a. $(8 + 4) \div 2 = (8 \div 2) + (4 \div 2)$
   b. $(16 + 8) \div 8 = (16 \div 8) + (8 \div 8)$
   c. $(18 + 8) \div 2 = (18 \div 2) + (8 \div 2)$
   d. $(25 + 20) \div 5 = (25 \div 5) + (20 \div 5)$
   e. $(1000 + 500) \div 500 = (1000 \div 500) + (500 \div 500)$

6. a, b, f,

7. Yes, but only if $a = 0$

Chapter 10

1. a. $40 \times 30 = 1200$ write the product
   b. $42 \times 30 = (40 + 2) \times 30$ $42 = 40 + 2$
   $= (40 \times 30) + (2 \times 30)$ distributive
   $= 1200 + 60$ write the product
   $= 1260$ addition
   c. $76 \times 80 = (70 + 6) \times 80$ $76 = 70 + 6$
   $= (70 \times 80) + (6 \times 80)$ distributive
   $= 5600 + 480$ write products
   $= 6080$ addition

d. $90 \times 57 = 90 \times (50 + 7)$ $57 = 50 + 7$
$= (90 \times 50) + (90 \times 7)$ distributive
$= 4500 + 630$ write products
$= 5130$ addition

e. $50 \times 76 = 50 \times (70 + 6)$ $76 = 70 + 6$
$= (50 \times 70) + (50 \times 6)$ distributive
$= 3500 + 300$ write products
$= 3800$ addition

f. $52 \times 47 = (50 + 2) \times 47$ $52 = 50 + 2$
$= (50 \times 47) + (2 \times 47)$ distributive
$= [50 \times (40 + 7)] + [2 \times (40 + 7)]$ $47 = 40 + 7$
$= (50 \times 40) + (50 \times 7) + (2 \times 40) + (2 \times 7)$ distributive
$= 2000 + 350 + 80 + 14$ write products
$= 2444$ addition

2. The 866 is in the hundreds' position and is meant to convey 200×433, not 2×433.

3. $4 \times 433 = 4 \times [(4 \times 100) + (3 \times 10) + 3]$
$= [4 \times (4 \times 100)] + [4 \times (3 \times 10)] + [4 \times 3]$
$= [(4 \times 4) \times 100] + [(4 \times 3) \times 10] + 12$
$= (16 \times 100) + (12 \times 10) + 12$
$= [(10 + 6) \times 100] + [(10 + 2) \times 10] + (10 + 2)$
$= (10 \times 100) + (6 \times 100) + (10 \times 10) + (2 \times 10) + 10 + 2$
$= (1 \times 1000) + (6 \times 100) + (1 \times 100) + (2 \times 10) + (1 \times 10) + 2$
$= 1000 + (7 \times 100) + (3 \times 10) + 2$
$= 1000 + 700 + 30 + 2$
$= 1732$

It is sufficient to show
$4 \times 433 = 4 \times (400 + 30 + 3)$
$= (4 \times 400) + (4 \times 30) + (4 \times 3)$
$= 1600 + 120 + 12$
$= 1732.$

4. $4 \times 433 = 433 \times 4$
$= 433 \times (1 + 1 + 1 + 1)$
$= (433 \times 1) + (433 \times 1) + (433 \times 1) + (433 \times 1)$
$= 433 + 433 + 433 + 433$

5. a.

$$\begin{array}{r} 47 \\ \underline{-\ 8} \\ 39 \\ \underline{-\ 8} \\ 31 \\ \underline{-\ 8} \\ 23 \\ \underline{-\ 8} \\ 15 \\ \underline{-\ 8} \\ 7 \end{array}$$

$47 = (5 \times 8) + 7$

b.

$$\begin{array}{r} 28 \\ \underline{-\ 7} \\ 21 \\ \underline{-\ 7} \\ 14 \\ \underline{-\ 7} \\ 7 \\ \underline{-\ 7} \\ 0 \end{array}$$

$28 = (4 \times 7) + 0$

6. 23 times

Chapter 11

1.
$$\begin{array}{r|l} 8\overline{)512} & \\ \underline{480} & 60 \\ 32 & \\ \underline{32} & \underline{4} \\ 0 & 64 \end{array}$$

$512 = (64 \times 8) + 0$

2.
$$\begin{array}{r|l} 7\overline{)644} & \\ \underline{630} & 90 \\ 14 & \\ \underline{14} & \underline{2} \\ 0 & 92 \end{array}$$

$644 = (92 \times 7) + 0$

3.
$$\begin{array}{r|l} 21\overline{)526} & \\ \underline{420} & 20 \\ 106 & \\ \underline{105} & \underline{5} \\ 1 & 25 \end{array}$$

$526 = (25 \times 21) + 1$

4.
$$\begin{array}{r|l} 18\overline{)779} & \\ \underline{720} & 40 \\ 59 & \\ \underline{54} & \underline{3} \\ 5 & 43 \end{array}$$

$779 = (43 \times 18) + 5$

5.
$$\begin{array}{r|l} 42\overline{)836} & \\ \underline{420} & 10 \\ 416 & \\ \underline{378} & \underline{9} \\ 38 & 19 \end{array}$$

$836 = (19 \times 42) + 38$

6.
$$\begin{array}{r|l} 23\overline{)14} & \\ \underline{0} & \underline{0} \\ 14 & 0 \end{array}$$

$14 = (0 \times 23) + 14$

7.
```
14/23  |
   14  | 1
   -------
    9    1
```

23 = (1 × 14) + 9

8.
```
19/720 |
   570 | 30
   150 |
   133 |  7
   ---------
    17   37
```

720 = (37 × 19) + 17

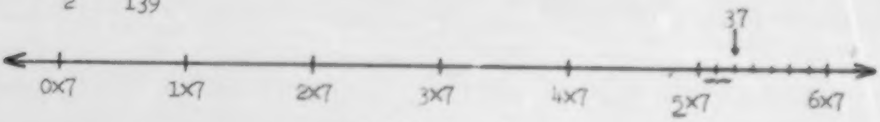
9.
```
100/50 |
     0 | 0
    ------
    50   0
```

50 = (0 × 100) + 50

10.
```
47/6535 |
   4700 | 100
   1835 |
   1410 |  30
    425 |
    423 |   9
   ----------
      2   139
```

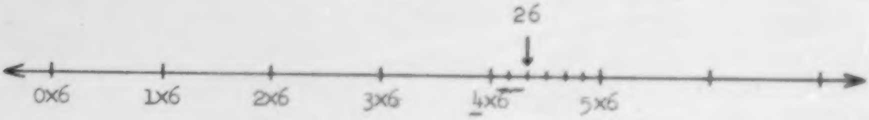
6535 = (139 × 47) + 2

11.


37
0×7
1×7
2×7
3×7
4×7
5×7
6×7

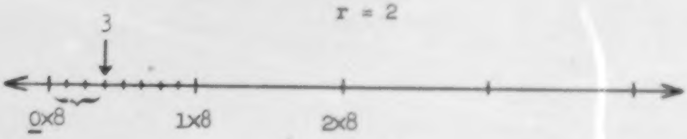
q = 5
r = 2

12.


26
0×6
1×6
2×6
3×6
4×6
5×6

q = 4
r = 2

13.



q = 0
r = 3

14. 0 = (0 × 52) + 0

Chapter 12

1. a. $n + 4 = 6$
   b. $4 + 3 = n$

2. a. 20 d. 29
   b. 0 e. 30
   c. 0 f. 12

3. $n + 12 = 15$
   $12 + n = 15$
   $n = 15 - 12$, etc.

4. a. Addition d. Subtraction
   b. Subtraction e. Subtraction
   c. Subtraction f. Subtraction

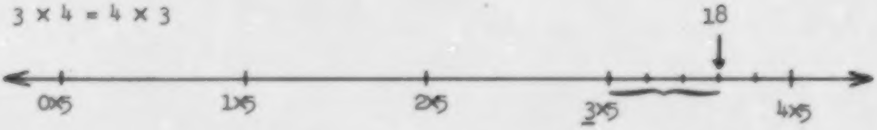
5. a. $8 > 6$
   b. $3 + 4 > 6$
   c. $(20 + 30) = (30 + 20)$
   d. $(200 + 800) > (200 + 700)$
   e. $(1200 + 1000) = (1000 + 1200)$

6. a. 33
   b. 140
   c. 60

7. c. $8 - 3 = 5$ f. $6 - n = 3$
   d. $8 + 10 = 18$ g. $4 + 2 = n$
   e. $45 - 20 = 25$ h. $q + n = p$

8. $3 \times 4 = 4 \times 3$

9.



$18 = (3 \times 5) + 3$

Chapter 13

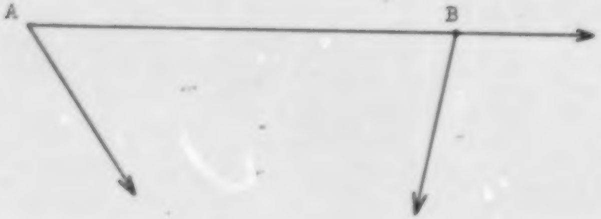
1. a. many
   b. one

2. a. many
   b. many
   c. one

3. a. one
   b. one line, many points

4. a. one
   b. one

5.



6. c

7. a

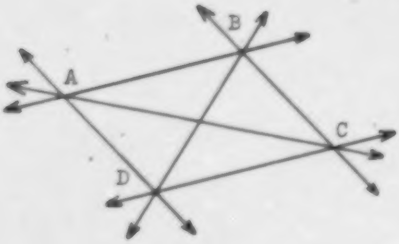
8. b

9. c

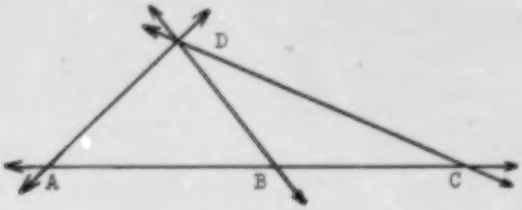
10. line

11. a. $\overrightarrow{AC}$; b. $\overrightarrow{CA}$; c. $\overline{AD}$; d. $\overleftrightarrow{AC}$

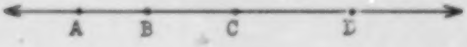
12. a. 6 if no two points lie in the same straight line.



4 if three points lie in one line.



1 if all four points lie in one line



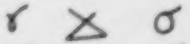
b. 6.

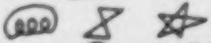
Chapter 14

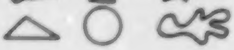
1. a, c, e, f.

2. The endpoints do not coincide

3. a, b, e

4. a. 

   b. 

   c. 

5. a, b, e, g, h

6. b, e.

7.

	Vertex	Sides
a.	O	$\overrightarrow{OA}$, $\overrightarrow{OB}$
b.	X	$\overrightarrow{XY}$, $\overrightarrow{XZ}$
c.	S	$\overrightarrow{SR}$, $\overrightarrow{ST}$
d.	Q	$\overrightarrow{QP}$, $\overrightarrow{QR}$

8. Interior

9. Interior

10. Exterior, Exterior

11. a. Yes. b. Yes. The interior is that part of the plane that contains the interior of $\triangle$ BAC.

12. P is in the interior of the second polygon but not the first.

13. Yes.

14. Yes

15. No

Chapter 15

1. $\overline{AB} \cong \overline{GH}$

2. a. $\overline{AB} > \overline{CD}$
   b. $\overline{AB} \cong \overline{CD}$
   c. $\overline{AB} > \overline{CD}$

3. b, d, g

4. $\angle DEF > \angle ABC$, $\angle DEF < \angle GHI$, $\angle DEF \cong \angle KLM$, $\angle DEF < \angle NOB$,
   $\angle DEF > \angle QRS$; $\angle DEF \cong \angle TVW$.
   $\angle GHI < \angle NOP$. $\angle NOP >$ all the other angles.

5.

	a. by sides	b. by angles
1.	isosceles	acute
2.	equilateral	acute
3.	scalene	right
4.	isosceles	obtuse
5.	scalene	acute
6.	scalene	obtuse
7.	scalene	right
8.	scalene	right
9.	scalene	acute
10.	isosceles	right

6. a. $\overline{OA}$ $\overline{OC}$, $\overline{OB}$, $\overline{OD}$
   b. $\overline{BC}$
   c. Interior
   d. no
   e. O, Z, D, A, B, C
   f. Yes
   g. $\overset{\frown}{AD}$, $\overset{\frown}{AC}$. $\overset{\frown}{ADB}$, $\overset{\frown}{ACB}$, $\overset{\frown}{ACD}$, $\overset{\frown}{ADC}$

7. a. Yes. b. Yes c. Yes d. No

8. a. A simple closed polygon is a simple closed curve made up of 3 or more line segments.
   b. A convex polygon is a polygon whose interior lies in the interior of each of its angles.
   c. An equilateral triangle is a triangle whose sides are all congruent.
   d. A circle is a simple closed curve with a point O in its interior such that if A and B are any two points on the curve $\overline{OA} \cong \overline{OB}$.

9. interior

10. The circle with radius $\overline{OA}$ lies in the interior of the circle with radius $\overline{OB}$.

11. 4

12. a. Two circles are congruent if a representation (tracing) of one can be matched with a representation (tracing) of the other. This is the general idea of congruence of any two geometric figures.
    b. Two circles are congruent if their radii are congruent.

13. The union is a segment which we could think of as the "perimeter" of the polygon. We will find in Chapter 16 that it is the "measure" of this segment which is the perimeter.

14. The center

Chapter 16

1. c, e

2. a. pounds; b. 18; c. 18 pounds

3. a. 9 chalk pieces; b. 9; c. a chalk piece

4. c, e

5. a. 170 mm.
   b. 340 cm.
   c. 4.8 cm.
   d. .357 m.
   e. .93 m.
   f. 9100 mm.

6. a. 1, 1, 1
   b. 45", 4 feet
   c. 4 is not the sum of 1, 1, 1.
   d. The error of the measure of each side was an excess. Even though each error was less than $\frac{1}{2}$ foot, the sum of the errors was over half a foot and therefore must be counted in the measure of the perimeter.

7. Each letter weighed $1\frac{1}{4}$ ounces requiring two 4¢ stamps. But all five letters weighted only 7 ounces requiring only 7 4¢ stamps. (Note that the problem was written before the 1963 raise in postage rates.)

8. 90

9. 60

10. 120

11. Each is 1. But $\overline{EF} > \overline{CD}$. The measure is assigned in terms of the unit AB. $EF > \overline{AB}$ and $\overline{AB} > \overline{CD}$, but the difference in each case ie less than $\frac{1}{2}\overline{AB}$ so the measure of each is 1.

12. $\overline{WZ} < \frac{1}{2}$ the unit segment. So its measure must be 0 in terms of this unit. Similarly $\angle MNO < \frac{1}{2}$ the unit angle. The units suggested are not appropriate to measure small segments or angles. It is like trying to measure the length of a desk in miles.

Chapter 17

1. 101, 103, 107, 109, 113, 127, 131, 137, 139, 149

2. a. 4 × 3
   b. 12 × 3, 18 × 2
   c. impossible
   d. impossible
   e. 4 × 2
   f. impossible
   g. 7 × 5
   h. impossible
   i. 3 × 13
   j. 7 × 6, 21 × 2, 14 × 3
   k. 3 × 2
   l. impossible
   m. 41 × 2
   n. 5 × 19

3. a. 3 × 5; b. 2 × 3 × 5; c. 3 × 3 × 5; d. 13 is a prime.

4. a. 3 × 5 × 7
   b. 2 × 2 × 3 × 5 × 5
   c. 2 × 2 × 2 × 2 × 2 × 2
   d. prime
   e. 2 × 2 × 2 × 5 × 5 × 5
   f. prime
   g. 17 × 19

5. a. 15
   b. 1
   c. 12
   d. 15
   e. 16
   f. 2
   g. 12
   h. 8
   i. 15
   j. 10

6. a. 6 (6 × 0 = 0 and 6 × 1 = 6)
   b. 1
   c. 1

7. a. Yes
   b. Yes
   c. Yes

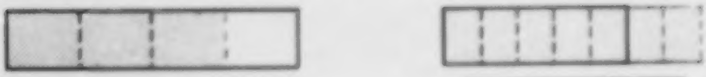
8. a. 6
   b. 15
   c. 21
   d. 35
   e. 30
   j. 90
   g. 26
   h. 77
   i. 2100
   j. 143
   k. 30
   l. 4800

9. a. 6
   b. 6
   c. 29
   d. 29
   e. a
   f. a

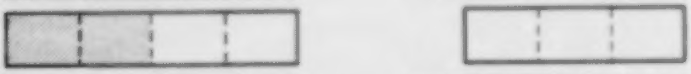
10. a. Yes, if they are the same number
    b. Yes. g.c.f. of 2 and 3 is 1, l.c.m. of 2 and 3 is 6
    c. No

Chapter 18

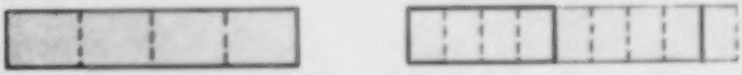
1. a.



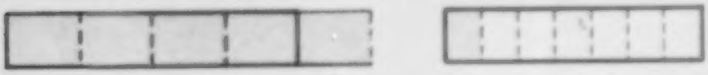
b.



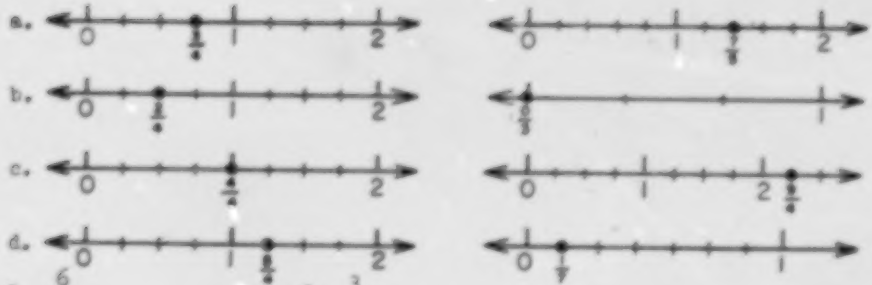
c.



d.



2.


a.
0
1
2
0
1
2
b.
0
1
2
1
c.
0
2
0
1
2
d.
0
1
2
0
1

3. a. $\frac{6}{4}$ f. $\frac{3}{2}$

b. $\frac{2}{8}$ g. not an appropriate model

c. $\frac{3}{8}$ h. not an appropriate model

d. $\frac{4}{6}$ i. not an appropriate model

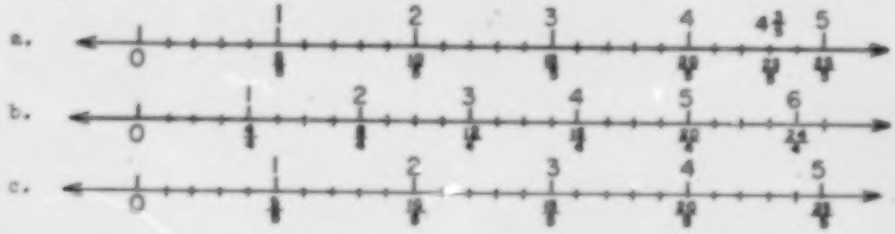
e. $\frac{6}{10}$ j. $\frac{6}{6}$

4. a. A, $\frac{1}{2}$ or $\frac{2}{4}$; B, $\frac{1}{4}$; C, $\frac{3}{4}$; D, $\frac{1}{3}$; E, $\frac{2}{3}$

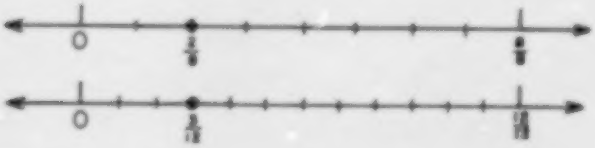
b. less than, since B lies to the left of D while 1 lies to the right of 0.

c. $\frac{1}{2}$ or $\frac{2}{4}$

5.


a.
0
1
2
3
4
$4\frac{1}{5}$
5
b.
0
1
2
3
4
5
6
c.
0
1
2
3
4
5

6.



Chapter 19

1. $\frac{11}{4}$, $\frac{7}{12}$, $\frac{12}{13}$, $\frac{7}{412}$, $\frac{412}{7}$, $\frac{2}{3}$

2. a. $\frac{11}{4}$, $\frac{7}{12}$, $\frac{12}{13}$, $\frac{7}{412}$, $\frac{412}{7}$, $\frac{2}{3}$ each have a prime number either in the numerator or the denominator and each fraction is in lowest terms. $\frac{13}{26}$ has a prime in the numerator but is not in lowest

terms since 26 is a multiple of 13.

b. If either the numerator (or denominator) is a prime, the only way the fraction can be reduced to simpler form is to divide numerator and denominator by that prime. But this will be possible only if the denominator (or numerator) is a multiple of that prime.

3. a. $\frac{1}{25} < \frac{1}{24}$ since $\frac{24}{25 \times 24} < \frac{25}{24 \times 25}$

b. $\frac{11}{24} < \frac{12}{26}$ since $\frac{11 \times 26}{24 \times 26} < \frac{12 \times 24}{26 \times 24}$ since $\frac{286}{24 \times 26} < \frac{288}{24 \times 26}$

c. $\frac{7}{8} > \frac{5}{6}$ since $\frac{7 \times 6}{8 \times 6} > \frac{8 \times 5}{6 \times 8}$

d. $\frac{17}{32} > \frac{1}{2}$ since $\frac{17}{32} > \frac{16}{32}$

e. $\frac{13}{26} = \frac{9}{18}$ since both are $= \frac{1}{2}$.

4. A. a. $\frac{3}{6} < \frac{5}{5}$ b. $\frac{4}{5} < \frac{10}{11}$ c. $\frac{15}{8} < \frac{25}{12}$

$3 \times 5 < 6 \times 3$ $4 \times 11 < 5 \times 10$ $15 \times 12 < 8 \times 25$

$15 < 18$ $44 < 50$ $180 < 200$

d. $\frac{9}{12} > \frac{8}{12}$ e. $\frac{13}{15} > \frac{2}{3}$ f. $\frac{337}{113} < \frac{167}{55}$

$9 \times 12 > 15 \times 8$ $13 \times 3 > 15 \times 2$ $337 \times 55 < 113 \times 167$

$108 > 96$ $39 > 30$ $18535 < 18871$

B. $\frac{a}{b} < \frac{c}{d}$ if $a \times d < b \times c$

$\frac{a}{b} > \frac{c}{d}$ if $a \times d > b \times c$

Chapter 20

1. a. $\frac{25}{30}$, $\frac{12}{30}$

b. $\frac{50}{60}$, $\frac{24}{60}$

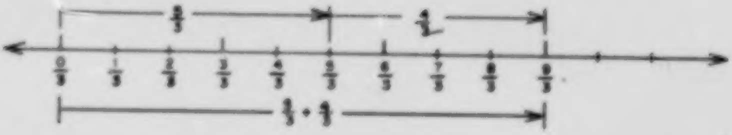
2. a. $\frac{14}{112}$; $\frac{48}{112}$

b. Impossible; the lowest common denominator is 56.

3. a. $\frac{45}{60}$, $\frac{12}{60}$, $\frac{50}{60}$

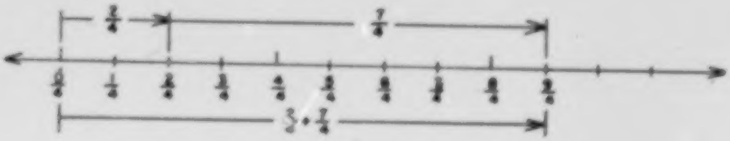
b. $\frac{20}{48}$, $\frac{9}{48}$, $\frac{32}{48}$

4. a.



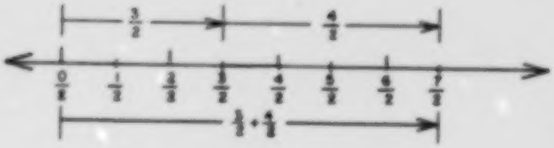
$\frac{5}{3}$
$\frac{4}{3}$
$\frac{0}{3}$ $\frac{1}{3}$ $\frac{2}{3}$ $\frac{3}{3}$ $\frac{4}{3}$ $\frac{5}{3}$ $\frac{6}{3}$ $\frac{7}{3}$ $\frac{8}{3}$ $\frac{9}{3}$
$\frac{5}{3} + \frac{4}{3}$

b.



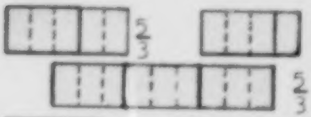
$\frac{2}{4}$
$\frac{7}{4}$
$\frac{0}{4}$ $\frac{1}{4}$ $\frac{2}{4}$ $\frac{3}{4}$ $\frac{4}{4}$ $\frac{5}{4}$ $\frac{6}{4}$ $\frac{7}{4}$ $\frac{8}{4}$ $\frac{9}{4}$
$\frac{2}{4} + \frac{7}{4}$

c.



$\frac{3}{2}$
$\frac{4}{2}$
$\frac{0}{2}$ $\frac{1}{2}$ $\frac{2}{2}$ $\frac{3}{2}$ $\frac{4}{2}$ $\frac{5}{2}$ $\frac{6}{2}$ $\frac{7}{2}$
$\frac{3}{2} + \frac{4}{2}$

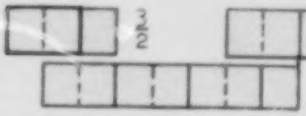
5. a.



$\frac{5}{3}$
$\frac{4}{3}$

$\frac{5}{3} + \frac{4}{3} = \frac{9}{3}$

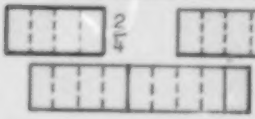
c.



$\frac{3}{2}$
$\frac{4}{2}$

$\frac{3}{2} + \frac{4}{2} = \frac{7}{2}$

b.



$\frac{2}{4}$
$\frac{7}{4}$

$\frac{2}{4} + \frac{7}{4} = \frac{9}{4}$

6. a. $\frac{3}{3} + \frac{5}{3}$

b. $\frac{7}{4} + \frac{5}{4}$

7. a. $\frac{7}{6} + \frac{5}{6} = 2$

b. $n + \frac{4}{4} = 2\frac{3}{4}$

c. $n + \frac{6}{8} = 2\frac{4}{8}$

8. a. $\frac{4}{2}$ or 2

b. $\frac{6}{2}$ or 3

c. No

d. No

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9. a. $1\frac{13}{24}$ d. $6\frac{19}{40}$

b. $11\frac{17}{24}$ e. $2\frac{11}{24}$

c. $11\frac{17}{24}$ f. $\frac{1}{24}$

Chapter 21

1. a. 5; $\frac{3}{4} \times 5 = \frac{15}{4}$

b. $\frac{5}{6}$; $\frac{1}{4} \times \frac{5}{6} = \frac{5}{24}$

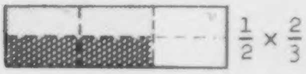
c. $\frac{7}{5}$; $\frac{4}{3} \times \frac{7}{5} = \frac{28}{15}$

2. a. $\overline{DE}$; b. $\overline{DI}$; c. $\overline{DI}$; d. $\overline{DF}$; e. $\overline{DE}$

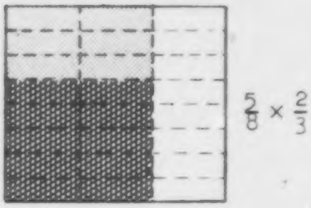
3. a. $\overline{AB}$; b. $\overline{AC}$; c. $\overline{AC}$; d. $\overline{AD}$; e. $\overline{AC}$

4. a. $\overline{KN}$; b. $\overline{KO}$; c. $\overline{KO}$

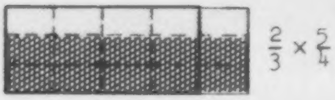
5. a.


$\frac{1}{2} \times \frac{2}{3}$

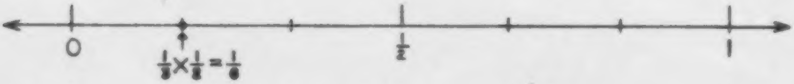
b.


$\frac{5}{8} \times \frac{2}{3}$

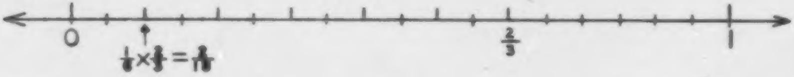
c.


$\frac{2}{3} \times \frac{5}{4}$

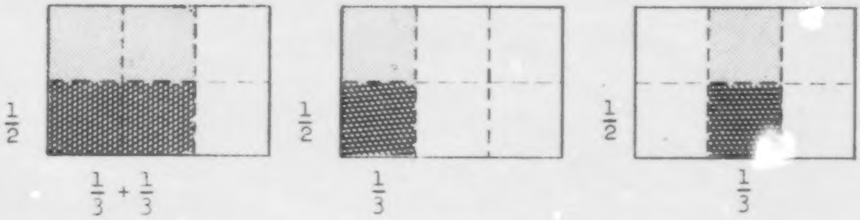
6. a.


0
$\frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$
$\frac{1}{2}$
1

b.


0
$\frac{1}{6} \times \frac{2}{3} = \frac{2}{18}$
$\frac{2}{3}$
1

7.


$\frac{1}{2}$
$\frac{1}{3} + \frac{1}{3}$
$\frac{1}{2}$
$\frac{1}{3}$
$\frac{1}{2}$
$\frac{1}{3}$

$\frac{1}{2} \times (\frac{1}{3} + \frac{1}{3})$ = $(\frac{1}{2} + \frac{1}{3})$ + $(\frac{1}{2} \times \frac{1}{3})$

8. a. $\frac{2}{3} \times \frac{15}{24} = \frac{2 \times 15}{3 \times 24} = \frac{2 \times (3 \times 5)}{3 \times (2 \times 2 \times 2 \times 3)} = \frac{2 \times 3 \times 5}{3 \times 2 \times 2 \times 2 \times 3}$

$= \frac{(2 \times 3) \times 5}{(2 \times 3) \times (2 \times 2 \times 3)} = \frac{2 \times 3}{2 \times 3} \times \frac{5}{2 \times 2 \times 3} = 1 \times \frac{5}{12} = \frac{5}{12}$

b. $\frac{7}{3} \times \frac{2}{12} \times \frac{6}{14} = \frac{7 \times 2 \times 6}{3 \times 12 \times 14} = \frac{7 \times 2 \times 2 \times 3}{3 \times 2 \times 2 \times 3 \times 2 \times 7}$

$= \frac{2 \times 2 \times 3 \times 7 \times 1}{2 \times 2 \times 3 \times 7 \times 2 \times 3} = \frac{2 \times 2 \times 3 \times 7}{2 \times 2 \times 3 \times 7} \times \frac{1}{2 \times 3}$

$= 1 \times \frac{1}{6} = \frac{1}{6}$

c. $\frac{12}{21} \times \frac{7}{16} = \frac{12 \times 7}{21 \times 16} = \frac{2 \times 2 \times 3 \times 7}{3 \times 7 \times 2 \times 2 \times 2 \times 2} = \frac{2 \times 2 \times 3 \times 7 \times 1}{2 \times 2 \times 3 \times 7 \times 2 \times 2}$

$= \frac{2 \times 2 \times 3 \times 7}{2 \times 2 \times 3 \times 7} \times \frac{1}{2 \times 2} = 1 \times \frac{1}{4} = \frac{1}{4}$

d. $\frac{54}{12} \times \frac{6}{24} = \frac{54 \times 6}{12 \times 24} = \frac{(2 \times 3 \times 3 \times 3) \times (2 \times 3)}{(2 \times 2 \times 3) \times (2 \times 2 \times 2 \times 3)}$

$= \frac{2 \times 2 \times 3 \times 3 \times 3 \times 3}{2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 3} = \frac{(2 \times 2 \times 3 \times 3) \times (3 \times 3)}{(2 \times 2 \times 3 \times 3) \times (2 \times 2 \times 2)}$

$= \frac{2 \times 2 \times 3 \times 3}{2 \times 2 \times 3 \times 3} \times \frac{3 \times 3}{2 \times 2 \times 2} = 1 \times \frac{9}{8} = \frac{9}{8}$

9. a. $\frac{6}{2} \times \frac{6}{5}$. 2 is a prime factor of denominator only once.

b. 2 is not a factor of the numerator.

c. The expression is not a fraction but the sum of two fractions. As it stands it is a sum not a division.

10. $R = \{1, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{7}{6}, \frac{8}{7}\}$

11. a. when the number is between 0 and 1 exclusive of 0 and 1.

b. when the number is greater than 1.

c. when the number is 1

d. Yes

Chapter 22

1. a. False, division is not commutative.

b. False, division is not associative.

c. False, division is not commutative.

d. False, division is not commutative.

e. True, 1 is a right hand identity.

f. True, right hand distributive property of division

g. False, there is no left hand distributive property of division.

h. False

i. True, $0 \div \frac{a}{b} = 0$ (if $\frac{a}{b} \neq 0$)

j. True, $0 \div \frac{a}{b} = 0$ (if $\frac{a}{b} \neq 0$)

k. True

l. True

m. True (closure)

n. False, division by 0 is meaningless.

o. True

2. a. 10; b. $\frac{16}{49}$

3. a. =   e. <

b. >   f. =

c. =   g. >

d. <   h. >

4. Two methods are outlined:

$$\frac{1}{\frac{5}{2}} = \frac{1 \times \frac{2}{5}}{\frac{5}{2} \times \frac{2}{5}} = \frac{\frac{2}{5}}{1} = \frac{2}{5} \quad \text{or} \quad \frac{1}{\frac{5}{2}} = \frac{1 \times \frac{2}{1}}{\frac{5}{2} \times \frac{2}{1}} = \frac{2}{\frac{5 \times 2}{1 \times 2}} = \frac{2}{\frac{5}{1} \times \frac{2}{2}} = \frac{2}{5 \times 1} = \frac{2}{5}$$

5. $$\frac{1}{\frac{a}{b}} = \frac{1 \times \frac{b}{a}}{\frac{a}{b} \times \frac{b}{a}} = \frac{\frac{b}{a}}{1} = \frac{b}{a} \quad \text{or} \quad \frac{1}{\frac{a}{b}} = \frac{1 \times \frac{b}{1}}{\frac{a}{b} \times \frac{b}{1}} = \frac{b}{\frac{a \times b}{1 \times b}} = \frac{b}{a \times \frac{b}{b}} = \frac{b}{a \times 1} = \frac{b}{a}$$

6. $2\frac{1}{2} = \frac{5}{2}$ while $\frac{1}{2\frac{1}{2}} = \frac{1}{\frac{5}{2}} = \frac{2}{5}$ by Exercise 4.

But $\frac{5}{2}$ and $\frac{2}{5}$ are reciprocals since $\frac{5}{2} \times \frac{2}{5} = \frac{5 \times 2}{2 \times 5} = 1$.

7. a. $\frac{1}{\frac{2}{3}} = \frac{3}{2}$ d. $\frac{1}{\frac{11}{9}} = \frac{9}{11}$

b. $\frac{1}{1\frac{1}{2}} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$ e. $\frac{1}{27\frac{1}{2}} = \frac{1}{\frac{55}{2}} = \frac{2}{55}$

c. $\frac{1}{22}$ f. $\frac{1}{\frac{1}{\frac{1}{2}}} = \frac{1}{2}$

8. a. True   e. True

b. True   f. True

c. False   g. True

d. False

9. a. $\frac{2}{3} \div \frac{5}{7} = \frac{\frac{2}{3}}{\frac{5}{7}} = \frac{\frac{2}{3} \times 3 \times 7}{\frac{5}{7} \times 3 \times 7} = \frac{(\frac{2}{3} \times 3) \times 7}{(\frac{5}{7} \times 7) \times 3} = \frac{2 \times 7}{5 \times 3} = \frac{14}{15}$

b. $\frac{3}{5} \div \frac{2}{3} = \frac{\frac{3}{5}}{\frac{2}{3}} = \frac{\frac{3}{5} \times 5 \times 3}{\frac{2}{3} \times 5 \times 3} = \frac{(\frac{3}{5} \times 5) \times 3}{(\frac{2}{3} \times 3) \times 5} = \frac{3 \times 3}{2 \times 5} = \frac{9}{10}$

10. $4\frac{1}{4} = \frac{5}{7} \times n \qquad n = \frac{17}{4} \div \frac{5}{7} = \frac{17}{4} \times \frac{7}{5} = \frac{17 \times 7}{4 \times 5} = \frac{119}{20} = 5\frac{19}{20}$

Answer: $5\frac{19}{20}$ hours.

11. $18{,}375 \div 25 = 18{,}375 \div \frac{100}{4} = 18{,}375 \times \frac{4}{100} = \frac{18{,}375 \times 4}{100} = \frac{73{,}500}{100} = 735$

12. $16 = \frac{2}{3} \times n \qquad n = \frac{16}{\frac{2}{3}} \qquad 16 \times \frac{3}{2} = 24$

13. a. yes, $\frac{8}{10}$

b. yes, $\frac{375}{1000}$

c. no

d. no

e. no

f. yes, $\frac{0}{10}$

g. yes, $\frac{6}{100}$

h. yes, $\frac{56}{1000}$

14. For non-zero numerators this is possible when only 2's and/or 5's appear as prime factors of the denominator.

Chapter 23

1. a.	three hundred	five tenths
b.	three hundredths	five thousandths
c.	three tenths	first 5 is five thousands; second 5 is five thousandths
d.	three thousandths	five tenths
e.	three hundredths	five thousandths
f.	three tenths	five

2. a. 375

b. 37.5

c. 3.75

d. .375

e. .0375

f. .00375

3. a. 1680 check $.6 \times 1680 = 1008.0$

   b. 57.04 check $3.75 \times 57.04 = 213.9$

   c. 95 check $6.8 \times 95 = 646$

   d. 11.9 check $11.9 \times 2.6 = 30.94$

4. a. $8.153 \div 2.63 = 3.1$

   b. $8.153 \div 26.3 = .31$

   c. $81.53 \div 26.3 = 3.1$

5. a. $.0078 \times 7.5 = .05850$

   b. $7.8 \times 7.5 = 58.50$

   c. $78 \times 7.5 = 585.0$

   d. $.075 \times 7,800 = 585.0$

6. $\frac{1}{13} = .\overline{076923}$

   a. no

   b. after 6 divisions since the remainder 1 reappears

   c. .076923

7. a. $\frac{2}{3} = .\overline{6}$ repeats with 6

   b. $\frac{7}{8} = .875$

   c. $\frac{1}{9} = .\overline{1}$ repeats with 1

8. a. $\frac{1}{11} = .\overline{09}$

   b. $\frac{4}{11} = .\overline{36}$

   c. $\frac{14}{11} = 1\frac{3}{11} = 1.\overline{27}$

9. Yes

10. a. .2, .4, .8

    b. .05, .15, .55

    c. .001, .111, .927

*11. a. $\frac{4}{11}$

    b. $\frac{1}{7}$

    c. 1 since $\frac{1}{3} = .\overline{3}$ and $\frac{1}{3} \times 3 = .\overline{3} \times 3 = .\overline{9}$

12. a. $\frac{163.8}{10.5} = \frac{1638}{105} = \frac{3 \times 7 \times 2 \times 3 \times 13}{3 \times 5 \times 7} = \frac{78}{5} = \frac{156}{10} = 15.6$

Answer: 15.6 miles per gallon.

b. $\frac{144}{50} = \frac{288}{100} = 2.88$ Answer 2.88 hours

c. $.60 + (7 \times .07) = .60 + .49 = 1.09$

$.48 + (7 \times .09) = .48 + .63 = 1.11$

Answer: Rosemary's was .02 pounds heavier

d. $2.3 \times 19.8 = 45.54$ Answer: 45.54 seconds

e. $50 \times 1.094 = 54.7$ Answer: 54.7 yards

Chapter 24

1. a. 3:5; b. 1:2; c. 3:2; d. 4:7; e. 8:10

2. a. 10:20 = 1:2; b. 2:4 = 1:2

3. a. 10:4 = 5:2; b. 4:10 = 2:5

c. 4:8 = 1:2; d. 8:4 = 2:1

4.

4:1	16:4	8:2	20:5	36:9	100:25	12:3	24:6	32:8	40:10	$1:\frac{1}{4}$

5.

a.	b.	c.
4:8	10:4	6:10
1:2	30:12	12:20
8:16	5:2	18:30
16:32	20:8	30:50
2:4	50:20	24:40
12:24	60:24	48:80
24:48	100:40	60:100
20:40	40:16	54.90
36:72	80:32	36:60
32:64	1000:400	42:70
3:6	15:6	3:5
$\frac{1}{2}$:1	25:10	9:15

6. 550

7. 200; 22 ; $\frac{1}{2}$; 80

Chapter 25

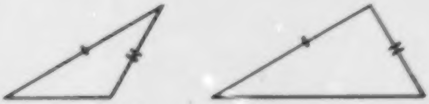
1. a. $\triangle BAD \cong \triangle DCB$
   b. $\triangle BGC \cong \triangle DGC$
   c. $\triangle ABC \cong \triangle FED$ $\triangle AFC \cong \triangle DCF$
   d. $\triangle ABD \cong \triangle CBD \cong \triangle CDB$
   e. $\triangle ABH \cong \triangle CDB \cong \triangle EFD \cong \triangle GHF$
   f. $\triangle ABE \cong \triangle CBE \cong CDE \cong \triangle ADE$; $\triangle ABD \cong \triangle CBD$; $\triangle ABC \cong \triangle ADC$

2. $\angle ABC \cong \angle E \cong \angle PQS \cong \angle ZYW$ and possibly $\angle D$

3. $\overline{AB} \cong \overline{PQ}$, $\overline{BC} \cong \overline{QR}$, $\overline{AC} \cong \overline{PR}$
   $\angle A \cong \angle P$, $\angle B \cong \angle Q$, $\angle C \cong \angle R$

4. Yes. $\triangle XYZ \cong \triangle MNL$

5. Not necessarily.



6. $m(\overline{PQ}) = 5$, $m(\angle PQR) = 42$

7. a. Yes
   b. No

8. a. $m(\angle A') = 30$, $m(\angle B') = 75$
   b. $m(\overline{A'C'}) = 8$
   c. $m(\overline{B'C'}) = 15$ $m(\overline{A'B'})$ is unknown
   d. $m(\overline{AC}) = 8$ $m(\overline{BC})$ is unknown

9. a and i
   d and h
   f and m

10. a. $m(\overline{PQ}) = 9$, $m(\overline{XZ}) = 15$
    b. $m(\overline{AC}) = 20$, $m(\overline{XZ}) = 25$

11. $\frac{17}{4}$ inches by $\frac{23}{4}$ inches

12. a. 85 miles
    b. $442\frac{1}{2}$ miles

Chapter 26

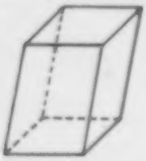
1. Triangular pyramid
2. Rectangular prism
3. Quadrangular pyramid

4. Rectangular prism
5. Sphere
6. Right prism
7. Right triangular prism
8. Right cylinder
9. Cone
10. 1. a. $\triangle BCD$; b. $\overline{AB}$; c. A; g. $\triangle ABC$
    2. a. A'B'C'D'; b. $\overline{AA'}$; c. A; f. $\overline{AA'}$; g. DCC'D'
    3. a. QRST; b. $\overline{PQ}$; c. Q; g. $\triangle PQR$
    4. a. x'y'z'w'; b. $\overline{XW}$; c. W; f. $\overline{XX'}$; g. WZZ'W'
    5. d. O; e. $\overline{OP}$; h. $\overline{PP'}$
    6. a. ABCDE; b. $\overline{AB}$; c. C; f. $\overline{DD'}$; g. CBB'C'
    7. a. $\triangle ABC$; b. $\overline{AD}$; c. B; f. $\overline{AD}$; g. BCFE
    8. a. circle ABC; f. $\overline{BD}$
    9. a. circle O; c. A; f. $\overline{VC}$

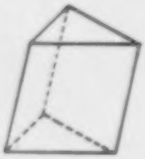
(The above are examples. Other answers are possible)

11. Circle of latitude is circle $\widehat{PQR}$.
    Circle of longitude is circle $\widehat{NPS}$.
    $\angle POD$ measures latitude of P.
    $\angle PTQ$ measures longitude of P.

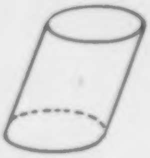
12. a.



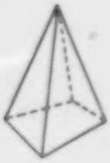
b.



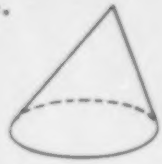
c.



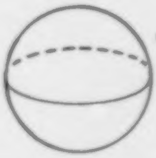
d.



e.



f.



Chapter 27

1. a
2. b
3. a
4. The triangle

5. a. 7
   b. 31
   c. 7, 31
   d. 24

6. a. 49
   b. 91
   c. 49, 91
   d. 42
   e. $10\frac{1}{2}$

7. a. $A = 5 \times 10$. Answer: 50 square feet
   b. $A = (2 \times 2) + (3 \times 2) = 10$. Answer: 10 square inches
   c. $A = \frac{1}{2} \times 60 \times 75 = 2250$. Answer: 225 square feet
   d. $A = (\frac{1}{2} \times 26 \times 48) + (\frac{1}{2} \times 26 \times 24) = (13 \times 48) + (13 \times 24)$
   $= 13 \times 72 = 936$. Answer: 936 square feet

8. $A = 183 \times 249 = 45{,}567$ Answer: 45,567 square inches
   Compare the answer to that of Problem 9.
   Answer: $316\frac{7}{16}$ square feet

9. length, 21 feet; width, 15 feet; area, 315 square feet
   The unit being larger, the measure of area is less accurate.

10. Estimate of area = 116 square units
    Estimate of $C = 37.5$ units
    Estimate of $d = 12$ units
    Estimate of $r = 6$ units
    $A = \frac{1}{2} \times C \times r = 3 \times 37.5 = 112.5$
    Estimate of area from the drawing is 3.5 units larger than the area computed from estimates of radius and circumference. This is surprisingly accurate.

11. $C = \pi \times d = 3.14 \times 4.2 = 13.188$
    $A = \pi \times r \times r = 3.14 \times 2.1 \times 2.1 = 13.8474$
    Answer: circumference is 13.2 inches
    area is 13.85 square inches

12. $A = \pi \times r \times r = 3.14 \times 144 = 452.16$ Answer: area is 452 square feet

Chapter 28

1. multiplied by two
2. multiplied by four
3. multiplied by eight
4. Yes, because in a right prism the lateral edge is perpendicular to the base.
5. No, since an edge is not perpendicular to the base as an altitude must be.
6. Too large
7. It is multiplied by two.
8. It is multiplied by eight.
9. It is multiplied by two.
10. It is multiplied by four.
11. $V = A \times h = (\frac{1}{2} \times r \times C) \times h = (\frac{1}{2} \times 6 \times 19) \times 12 = 57 \times 12$
    $V = 684$ **Answer:** 684 cubic inches
12. $V = \frac{1}{3} \times A \times h = \frac{1}{3} \times 684$ **Answer:** 228 cubic inches
13. $V = \frac{2}{3} \times 684 = 456$ **Answer:** 456 cubic inches
14. Surface area of cylinder $= (2 \times A) + (C \times h) = 2 \times (\frac{1}{2} \times r \times C) + (C \times h)$
    $= (6 \times 19) + (19 \times 12)$
    $= 19 \times (6 + 12) = 19 \times 18$
    $= 342$
    Answer: 342 square inches

    Surface area of sphere $= C \times 2 \times r = 19 \times 12 = 228$
    Answer: 228 square inches
15. a. r is 3 inches
    $V = \frac{4}{3} \times \pi \times 3 \times 3 \times 3 = 36\pi = 36 \times \frac{22}{7} = \frac{792}{7}$
    $V = 113.$ **Answer:** Volume is 113 cubic inches
    $A = 4 \times \pi \times 3 \times 3 = 36\pi = 113$ **Answer:** Area is 113 square inches

    b. r is 4000 miles
    $V = \frac{4}{3} \times \pi \times 4000 \times 4000 \times 4000$
    $A = 4 \times \pi \times 4000 \times 4000.$ The appropriate value of π depends on whether 4000 is a measure of r to the nearest mile or

whether it is to the nearest 10 miles or 100 miles or 1000 miles. Taking it to be to the nearest 100 miles means two figure accuracy on r and we can use 3.142 as an approximation to π. This gives $V = 258{,}116{,}000{,}000$ and $A = 201{,}090{,}000$. The radius of the earth is about 4000 miles. Its surface area is, therefore, about 201,090,000 square miles and its volume about 258,116,000,000 cubic miles.

c. $r = .01$ inch

$V = \frac{4}{3} \times \pi \times .01 \times .01 \times .01 = \frac{4 \times \pi \times .000001}{3}$

$V = .000004$ Answer: .000004 cubic inches

$A = 4 \times \pi \times .01 \times .01 = \pi \times .0004 = .0013$

Answer: .0013 square inches

Chapter 29

1. a. $^+3 < {}^+5$
   b. $\frac{^-12}{17} > \frac{-4}{5}$
   c. $\frac{-8}{3} < \frac{+6}{4}$
   d. $^+1 > {}^-19$
   e. $^-16 > {}^-32$
   f. $^+479 > {}^+421$
   g. $^+89 < {}^+95$
   h. $^-26 = {}^-26$
   i. $\frac{-3}{7} < \frac{-5}{12}$
   j. $0 > {}^-7$

2. a. $^-9$
   b. $^-7$
   c. 4
   d. $\frac{-17}{3}$
   e. 6
   f. 7

3. a. $^-2$ to $^+6$
   b. $^+8$ to $^-1$
   c. $^-6$ to 0
   d. $^+5$ to $^-3$
   e. $^-4$ to $^-8$

4. a. $^-7$
   b. $\frac{+22}{15}$
   c. $^-2$
   d. $^-5$
   e. $^+7$
   f. $^+12$
   g. $^-1$
   h. $^-4$

5. a. $^{-}23$
b. $^{-}24$
c. $^{+}35$
d. $^{-}16$
e. $^{+}8$
f. $^{-}9$
g. $^{+}3$
h. $^{-}3$

6. a. True
b. True
c. False
d. True
e. False
f. True

7. a. positive
b. negative
c. negative
d. positive
e. positive
f. negative

8. a. $^{+}7 + {}^{-}4 = {}^{-}4 + {}^{+}7 = 3$
b. $^{-}6 + {}^{-}12 = {}^{-}12 + {}^{-}6 = {}^{-}18$
c. $^{+}3 + {}^{+}11 = {}^{+}11 + {}^{+}3 = 14$
d. $^{-}7 + {}^{-}16 = {}^{-}16 + {}^{-}7 = {}^{-}23$
e. $^{-}13 + {}^{+}8 = {}^{+}8 + {}^{-}13 = {}^{-}5$
f. $^{-}6 + {}^{-}9 = {}^{-}9 + {}^{-}6 = {}^{-}15$
g. $^{+}10 + {}^{-}5 = {}^{-}5 + {}^{+}10 = {}^{+}5$
h. $^{+}32 + {}^{-}19 = {}^{-}19 + {}^{+}32 = {}^{+}13$

9. a. $^{-}3 + {}^{-}6 = {}^{-}6 + {}^{-}3$
b. $^{+}3 + {}^{-}6 < {}^{-}3 + {}^{+}6$
c. $^{+}6 + {}^{+}3 > {}^{-}6 + {}^{-}3$
d. $^{-}6 + {}^{+}3 = {}^{+}3 + {}^{-}6$
e. $^{+}7 + {}^{-}2 < {}^{+}7 + {}^{+}2$
f. $^{+}2 + {}^{-}7 = {}^{-}7 + {}^{+}2$
g. $^{-}2 + {}^{-}7 = {}^{-}2 + {}^{-}7$
h. $^{-}2 + {}^{+}7 > {}^{-}7 + {}^{+}2$

10. a. True
b. True
c. False
d. False
e. False
f. False
g. False
h. True

11. a. $^{-}12$
b. $\frac{2}{5}$
c. 1
d. $^{-}1$
e. 0
f. $^{-}36$
g. $\frac{4}{5}$
h. 1

12. a. $\frac{-2}{5}$
b. $\frac{-9}{16}$
c. $^{+}1$
d. 0
e. $\frac{4}{5}$
f. 1
g. 0
h. $^{-}4$

Chapter 30

1. a. $\sqrt{2}$ f. $\sqrt{5} + \sqrt{2}$

   b. $\frac{3 - \sqrt{2}}{7}$ g. $\sqrt{7}$

   e. $9 - \sqrt{3}$

2. Irrational

3. $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ It is irrational.

4. Yes, since $\frac{\sqrt{3}}{3} \times \sqrt{3} = \frac{\sqrt{3} \times \sqrt{3}}{3} = \frac{3}{3} = 1$

5. No. $\sqrt{2} \times \sqrt{2} = 2$

6. a. ${}^{-}\sqrt{2}$ d. ${}^{-}\sqrt{5} + {}^{-}\sqrt{2}$

   b. $\frac{-1}{\sqrt{2}}$ e. ${}^{-}\sqrt{7}$

   c. $\sqrt{2} - \sqrt{3}$ f. $\sqrt{5}$

7. No. $\sqrt{7} + {}^{-}\sqrt{7} = 0$

8. Irrational as is any non-repeating decimal.

9. Irrational as is any non-repeating decimal.

10. If $\underline{r}$ is a non-zero rational and $\underline{s}$ is an irrational and $r \times s = t$ then $\underline{t}$ must be irrational.

    If $\underline{t}$ were rational, then since $s = t \div r$, $\underline{s}$ would have to be rational because the rational numbers are closed under division by any non-zero number. But $\underline{s}$ is given to be irrational, so $\underline{t}$ must be irrational.

11. The answer is "no". π is irrational. See 10 above.

12. $A = \pi \times r \times r = 9 \times \pi$. But by Exercise 11 the product of the rational number 9 by the irrational number π is irrational.

GLOSSARY

Mathematical terms and expressions are frequently used with different meanings and connotations in the different fields or levels of mathematics. The following glossary explains some of the mathematical words and phrases as they are used in this book. These are not intended to be formal definitions. More explanations as well as figures and examples may be found in the book by reference through the index.

A

ADDITION. An operation on two numbers called the addends to obtain a third number called the sum.

ALGORITHM (ALGORISM). A numerical process that may be applied to obtain the solution of a problem.

ANGLE. The union of two rays which have the same endpoint but which do not lie in the same line.

ARC. A part of a circle determined by two points on the circle.

AREA. A numerical measure in terms of a specified unit which is assigned to a surface or a plane region. Note that both number and unit must be given, as 30 square feet.

ARRAY. An orderly arrangement of rows and columns which may be used as a physical model to interpret multiplication of whole numbers.

ASSOCIATIVE PROPERTY OF ADDITION. When three numbers are added in a stated order the sum is independent of the grouping, i.e., $(a + b) + c = a + (b + c)$.

ASSOCIATIVE PROPERTY OF MULTIPLICATION. When three numbers are multiplied in a stated order the product is independent of the grouping, i.e., $(a \times b) \times c = a \times (b \times c)$.

AXIOM (Syn. POSTULATE). A statement which is accepted without proof.

B

BASE (of a numeration system). The number used in the fundamental grouping. Thus 10 is the base of the decimal system and 2 is the base of the binary system.

BASE (of a geometric figure). A particular side or face of a geometric figure.

BINARY NUMERATION SYSTEM. A numeration system whose base is two.

BINARY OPERATION. An operation applied to a pair of numbers.

BISECT. To divide a segment or an angle into two congruent parts.

BRACES { }. Symbols used in this book exclusively to indicate sets of objects. The members of the set are listed or specified within the braces.

BRACKETS [ ]. Symbols used to indicate that the enclosed numerals or symbols belong together.

BROKEN LINE CURVE. A curve formed from segments joined end to end but not forming a straight line.

C

CIRCLE. The set of all points in a plane which are the same distance from a given point. Alternatively, a simple closed curve having a point O in its interior and such that if A and B are any two points of the circle $\overline{OA} \cong \overline{OB}$.

CIRCULAR REGION. The union of a circle and its interior.

CLOSED CURVE. A curve which has no endpoints; i.e., in drawing a representation the starting and end points are the same.

CLOSURE. An operation in a set has the property of closure if the result of the operation on members of the set is a member of the set. Thus addition and multiplication have closure in the set of whole numbers but subtraction and division do not. Division has closure in the set of positive rational numbers and subtraction has closure in the set of all rational numbers.

COMMON DENOMINATOR. A common multiple of the denominators of two or more fractions.

COMMUTATIVE PROPERTY. An operation is commutative if the result of using it with the ordered pair (a,b) is the same as with (b,a).

COMPOSITE NUMBER. A whole number greater than 1 which is not a prime number.

CONE. A cone is a set of points with a base which is a surface consisting of a plane region bounded by a simple closed curve, a point called the vertex not in the plane of the base, and all the line segments with one endpoint the vertex and the other any point in the given curve.

CONGRUENCE. The relationship between two geometric figures which have exactly the same size and shape.

CONVEX POLYGON. A polygon whose interior is in the interior of each of its angles.

CORRESPONDING ANGLES. Pairs of angles whose vertices are paired in a 1-1 pairing of the vertices of two polygons.

CORRESPONDING SIDES. Pairs of sides whose endpoints are paired in a 1-1 pairing of the vertices of two polygons.

COUNTING NUMBERS. Whole numbers with the exception of zero.

CUBE. A prism with square bases and square lateral faces.

CURVE. A set of all those points which lie on a particular path from A to B.

CYLINDER. A surface with bases which are congruent simple closed curves lying in parallel planes and with a lateral surface made up of parallel segments whose endpoints are in the curves.

D

DECAGON. A polygon with ten sides.

DECIMAL. A numeral written in the extended decimal place value system.

DECIMAL PLACE VALUE SYSTEM. A place value numeration system with ten as the base for grouping.

DECIMAL POINT. A dot written to indicate the units position in a decimal.

DEGREE. The most common unit for numerical measure of angles. The symbol for a degree is $^{\circ}$.

DENSE. A property of the sets of rational and real numbers. The rational (real) numbers are dense because between any two rational (real) numbers there is a third rational (real) number.

DIAMETER OF A CIRCLE. A line segment which contains the center of the circle and whose endpoints lie on the circle.

DIAMETER OF A SPHERE. A line segment which contains the center of the sphere and whose endpoints lie on the sphere.

DISJOINT SETS. Two or more sets which have no members in common.

DISTRIBUTIVE PROPERTY. A joint property of multiplication and addition. This property says that multiplication is distributive over addition which means that $a \times (b + c) = (a \times b) + (a \times c)$.

DIVISION. An operation on two numbers a and b such that $a \div b = n$ if and only if $n \times b = a$.

E

EDGE. The intersection of two polygonal regions which are faces of the surface of a solid.

ELEMENT OF A SET. An object in a set; a member of a set.

ELEMENT OF A CONE. Any segment from the vertex to a point in the boundary of the base.

ELEMENT OF A CYLINDER (PRISM). Any segment connecting corresponding points in the boundaries of the bases.

ELLIPSE. One of the curves determined by the intersection of the lateral surface of a cone with a plane.

EMPTY SET. The set which has no members.

EQUAL, symbol =. $A = B$ means that A and B are two different names for the same object. For example, $5 - 2 = 3$; and $A = B$ where $A = \{x, y, z\}$ and $B = \{z, x, y\}$.

EQUIVALENCE. A relationship existing between different numerals that name the same number.

EXPANDED FORM. 532 written as $(5 \times [10 \times 10]) + (3 \times 10) + (2 \times 1)$ is said to be written in expanded form.

EXTENDED DECIMAL PLACE VALUE SYSTEM. A decimal place value system extended so that places to the right of the decimal point indicate tenths, hundredths, thousandths, etc.

F

FACTOR. Any of the numbers to be multiplied to form a product.

FRACTION. Any expression of the form $\frac{x}{y}$ where x and y represent any numbers.

G

GREATER THAN, FOR NUMBERS. $\underline{a}$ is greater than $\underline{b}$ if $a - b$ is a positive number. It is written $a > b$.

GREATEST COMMON FACTOR. The largest whole number which is a factor of two or more given whole numbers.

H

HALF-LINE. A line separated by a point results in two half-lines, neither of which contains the point.

HALF-PLANE. A plane separated by a line results in two half-planes, neither of which contains the line.

HALF-SPACE. Space separated by a plane results in two half spaces, neither of which contains the plane.

HEXAGON. A polygon with six sides.

HYPERBOLIC ARC. One of the curves determined by the intersection of a plane with the lateral surface of a cone.

I

IDENTITY ELEMENT. For addition, is that number 0 such that $0 + a = a + 0 = a$; for multiplication, is that number 1 such that $a \times 1 = 1 \times a = a$.

INTEGER. Any whole number or its opposite.

INTERIOR OF A PLANE GEOMETRIC FIGURE. One of the sets of points into which the figure separates the plane in which it lies.

INTERIOR OF A SOLID GEOMETRIC FIGURE. One of the sets of points into which the figure separates the space in which it lies.

INTERSECTION. The set of points common to two or more sets of points.

INVERSE OPERATIONS. Two operations such that one "undoes" what the other one "does."

IRRATIONAL NUMBER. Any number which cannot be expressed in the form $\frac{a}{b}$ where a is an integer and b is a counting number, i.e., any real number that is not a rational number.

J

JOIN. The union of two disjoint sets.

L

LATERAL SURFACE. The surface of a prism, pyramid, cylinder or cone, exclusive of the bases.

LATITUDE. A line or circle of latitude is the intersection of the surface of the earth with a plane perpendicular to the line from the north to the south pole. Also a number assigned to such a line.

LEAST COMMON MULTIPLE. The smallest non-zero whole number which is a multiple of each of two given whole numbers.

LENGTH OF A LINE SEGMENT. A numerical measure in terms of a specified unit which is assigned to the segment. Note that both number and unit must be given, as 3 feet or 5 miles, etc.

LESS THAN, FOR NUMBERS. a is less than b if $b - a$ is a positive number. It is written $a < b$.

LESS THAN, FOR SETS. "A is less than B" means that in pairing elements of A with those of B, there are elements of B left over after all the elements of A have been paired.

LINE (STRAIGHT LINE). A particular set of points. Informally it can be thought as the extension of a line segment.

LINE SEGMENT. A special case of the curves between two points. It may be represented by a string stretched tautly between its two endpoints.

LONGITUDE. A line of longitude is the line of intersection of the surface of the earth with a plane passing through the north and south poles. Also the number assigned to such a line.

M

MATCH. Two sets match each other if their members can be put in one-to-one correspondence.

MEASURE. A number assigned to a geometric figure indicating its size with respect to a specific unit.

MEASURE, ERROR OF. Difference between an indicated measure of length (area, volume) and the number which is approximated better and better as the unit of measure becomes smaller and smaller.

MEMBER OF A SET. An object in a set.

MERIDIAN. Technical name for line of longitude.

METRIC SYSTEM. A decimal system of measure with the meter as the standard unit of length.

MORE THAN, FOR SETS. A is more than B if B is less than A. See LESS THAN.

MULTIPLE OF A WHOLE NUMBER. A product of that number and any whole number.

MULTIPLICATION. An operation on two numbers called factors to obtain a third number called the product.

N

NEGATIVE RATIONAL NUMBER. Any rational number less than zero.

NON-NEGATIVE RATIONAL NUMBER. All the positive rational numbers and zero.

NUMBER.

See Whole number
Counting number
Rational number
Negative rational number
Irrational number
Real number.

NUMBER LINE. A model to show numbers and their properties. The model is used first for the whole numbers. The markings and names are extended as the number system is extended until finally a 1-1 correspondence is set up between all the points of the line and all the real numbers.

NUMBER PROPERTY OF A SET. The number of elements in the set. The number property of set A is written N(A).

NUMERAL. A name used for a number.

NUMERATION SYSTEM. A numeral system for naming numbers.

NUMBER SENTENCE. A sentence involving numbers.

O

OCTAGON. A polygon with eight sides.

ONE-TO-ONE CORRESPONDENCE. A pairing between two sets A, B, which associates with each member of A a single member of B, and with each member of B a single member of A.

OPEN SENTENCE. A sentence with one or more symbols that may be replaced by the elements of a given set.

OPERATION. A (binary) operation is an association of an ordered pair of numbers with a third number.

OPPOSITE NUMBERS. A pair of numbers whose sum is 0.

ORDER. A property of a set of numbers which permits one to say when a and b are in the set whether a is "less than," "greater than," or "equal to" b.

ORDERED PAIR. An ordered pair of objects is a set of two objects in which one of them is specified as being first.

P

PAIRING. A correspondence between an element of one set and an element of another set.

PARABOLIC ARC. The intersection of the lateral surface of a cone with a plane which is parallel to an element.

PARALLEL LINES. Lines in the same plane which do not intersect.

PARALLEL PLANES. Planes that do not intersect.

PARALLELOGRAM. A quadrilateral whose opposite sides are parallel.

PARENTHESES ( ). Marks to indicate grouping.

PERCENT. Means "per hundred," as 3 per hundred or 3 percent.

PERIMETER. The length of the line segment which is the union of all the non-overlapping line segments congruent to the sides of the polygon.

PENTAGON. A polygon with five sides.

PLACE VALUE. The value given to a certain position in a numeral.

PLACE VALUE NUMERATION SYSTEM. A numeration system which uses the position or place in the numeral to indicate the value of the digit in that place.

PLANE. A particular set of points which can be thought of as the extension of a flat surface such as a table.

PLANE CURVE. A plane curve is a curve all points of which lie in a plane.

PLANE REGION. The interior of any simple closed plane curve together with the curve.

POINT. An undefined term. It may be thought of as an exact location in space.

POLYGON. A simple closed curve which is the union of three or more line segments.

POSITIVE RATIONAL NUMBER. Any rational number greater than zero.

PRIME NUMBER. Any whole number that has exactly two different factors (namely itself and 1).

PRISM. A surface consisting of the following set of points: two congruent polygonal regions which lie in parallel planes; and a number of other plane regions which are all bounded by parallelograms.

PROPORTION. A statement of equality between two ratios.

PYRAMID. A surface which is a set of points consisting of a polygonal region called the base, a point called the vertex not in the same plane as the base, and all the triangular regions determined by the vertex and the sides of the base.

Q

QUADRANGLE. A quadrilateral.

QUADRILATERAL. A polygon with four sides.

R

RADIUS OF CIRCLE. A line segment with one endpoint the center of the circle and the other endpoint on the circle.

RADIUS OF SPHERE. A line segment with one endpoint the center of the sphere and the other endpoint on the sphere.

RATE. A special kind of ratio with the comparison usually between two quantities of different types. Thus postage rates are cents per ounce; speed may be miles per hour, etc.

RATIO. A relationship $a:b$ between an ordered pair of numbers $\underline{a}$ and $\underline{b}$ where $b \neq 0$. The ratio may be also expressed by the fraction $\frac{a}{b}$.

RATIONAL NUMBER. In Chapters 18-28 a number which can be written in the form $\frac{a}{b}$ where $\underline{a}$ is a whole number and $\underline{b}$ is a counting number. In Chapters 29 and 30 such a number is called a positive rational number if it is not equal to zero. Rational numbers are then all the above numbers along with their opposites.

RAY, $\overrightarrow{AB}$. The union of a point A and all those points of the line $\overleftrightarrow{AB}$ on the same side of A as B.

REAL NUMBERS. The union of the set of rational numbers and the set of irrational numbers.

RECIPROCALS. Any pair of numbers whose product is 1.

RECTANGLE. A parallelogram with four right angles.

RECTANGULAR PRISM. A right prism whose base is a rectangle.

REGION. See PLANE REGION and SOLID REGION.

REGROUPING. A word used to replace the words "carrying" and "borrowing."

REMAINDER SET. If B is a subset of A a new set A - B is the remainder set. It consists of all the elements of A which are not elements of B.

S

SCALE. A scale is a number line with the segment from $\underline{0}$ to $\underline{1}$ congruent to the unit being used.

SEGMENT. See LINE SEGMENT.

SEPARATE. To divide a given set of points such as a line, plane, sphere, space, etc. into disjoint subsets by use of another subset such as a point, line, circle, plane, etc.

SET. A set is any collection of things listed or specified well enough so that one can say exactly whether a certain thing does or does not belong to it.

SIMPLE CLOSED CURVE. A plane closed curve which does not intersect itself.

SIMILAR. A relationship between two geometric figures which have the same shape but not necessarily the same size.

SKEW. Two lines which do not intersect and are not parallel.

SOLID REGION. All the points of the interior of a closed surface together with the points of the surface.

SOLUTION SET. The set of all numbers which make an open number sentence true.

SPACE. The set of all points.

SPHERE. The set of all points in space which are at the same distance from a given fixed point. Alternatively, a simple closed surface having a point O in its interior and such that if A and B are any two points in the surface $\overline{OA} \cong \overline{OB}$.

SUBSCRIPT. A symbol written at the lower right of another symbol.

SUBSET. Given two sets A and B, B is a subset of A if every member of B is also a member of A.

SUBTRACTION. The operation $a - b$ such that $a - b = n$ if $n + b = a$.

SUPERSCRIPT. A symbol written at the upper left or right of another symbol.

SURFACE AREA. The total area of the surface of a solid.

SURFACE OF A SOLID. A closed surface; loosely, the "skin" of the solid.

T

TALLY. A mark made to record each successive member of a set.

THEOREM. A statement provable on the basis of previously proved or assumed statements.

TRIANGLE. A polygon with three sides.

U

UNION (OF SETS A AND B). The set which has as its members all the members of A and also all the members of B, and no other members.

UNIQUE. An adjective meaning one and only one.

V

VERTEX (pl. VERTICES).

of an angle: the common endpoint of its two rays.

of a polygon: the common endpoint of two segments.

of a prism or pyramid: the common endpoint of three or more edges.

VOLUME. A numerical measure in terms of a specified unit which is assigned to a solid region. Note that both number and unit must be given, as 3 cubic feet.

W

WHOLE NUMBER. The common property associated with a set of matched sets.

Z

ZERO. The number associated with the empty set.

MATHEMATICAL SYMBOLS USED IN THIS TEXT

+	plus; add; also used as a superscript, e.g., $^{+}3$ for "positive three"
-	minus; subtract; also used as a superscript, e.g., $^{-}3$ for "negative three"
±	plus or minus
×	multiply
÷	divide
$\overline{)}$	division
$\sqrt{}$	square root
$\frac{a}{b}$	fraction; rational number; ratio; divide
∠, ∡	angle
△	triangle
$\overline{AB}$	line segment
$.\overline{23}$	a repeating decimal
$\widehat{AB}$	arc
=	is equal to
≅	is congruent to
<	is less than
≤	is less than or equal to
>	is greater than
≥	is greater than or equal to
≠	is not equal to
$\overline{B}$	B bar
∪	join or union
A'	A prime
π	pi (π = 3.14159 ...)
$\overleftrightarrow{AB}$	line
$\overrightarrow{AB}$	ray
N(A)	number property of set A
~	wiggle, removal of one set from another
{ }	curly braces
( )	parentheses
[ ]	brackets
:	a:b ratio
m($\overline{AB}$)	measure of a line segment
m(∠ ABC)	measure of an angle
30°	30 degrees

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